# Krea - RAM - Maths Circle - Session 10

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## 1 Overview

Building on Hilbert's Hotel, we develop a rigorous notion of "size" for infinite sets using bijections (one-to-one and onto functions), and explore whether all infinities are equal.

## 2 What Does "Same Size" Mean?

1. **Counting finite sets without numbers**
   - Suppose you have a pile of forks and a pile of knives, but you cannot count
   - How would you determine if there are the same number of each?
   - Key idea: Pair them up! If nothing is left over, they're the same size
2. **Formalizing the pairing idea**
   - A **bijection** between sets $A$ and $B$ is a rule that pairs each element of $A$ with exactly one element of $B$, and vice versa
   - Two sets have the **same cardinality** if there exists a bijection between them
   - We write $|A| = |B|$ when such a bijection exists

## 3 Comparing Infinite Sets

1. **Are there as many even numbers as natural numbers?**
   - $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ and $E = \{2, 4, 6, 8, 10, \ldots\}$
   - Find a bijection $f : \mathbb{N} \to E$
   - What is $f(n)$? Verify it pairs every natural number with exactly one even number

2. **Are there as many integers as natural numbers?**
   - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ seems "twice as big" plus zero
   - Can you list the integers in a sequence: first, second, third, ...?
   - Hint: Alternate between positive and negative: $0, 1, -1, 2, -2, 3, -3, \ldots$
3. **Are there as many rational numbers as natural numbers?**
   - $\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\}$ seems vastly larger
   - Between any two rationals, there are infinitely many more rationals!
   - Yet we can list them all: arrange in a grid and traverse diagonally

# 4 Student Activities and Discussion

Students worked through constructing explicit bijections for $\mathbb{N} \to E$ and $\mathbb{N} \to \mathbb{Z}$. They came up with the diagonal enumeration of $\mathbb{Q}$ and then attempted to find a bijection for $\mathbb{R}$ or give a reason as to why no such bijection exists. Will be expanded on next session.

# 5 Takeaway

The notion of bijection gives us a rigorous way to compare infinite sets. While many infinite sets (naturals, integers, rationals) are "countably infinite" with cardinality $\aleph_0$, the real numbers form an "uncountable" infinity, fundamentally larger.