# Basic Set Theory

A **set** is a collection of distinct objects, called **elements**. Sets are usually denoted by capital letters, and elements by lowercase letters.

$$A = \{1, 2, 3\}, \quad B = \{a, b, c\}$$

If $x$ is an element of $A$, we write $x \in A$. If $x$ is not an element of $A$, we write $x \notin A$.

## Describing Sets

There are two common ways to describe a set:

- **Roster form:** listing all elements

$$A = \{2, 4, 6, 8\}$$

- **Set-builder notation:**

$$A = \{x \mid x \text{ is an even natural number less than } 10\}$$

## Special Sets

- The **empty set** contains no elements and is denoted by $\varnothing$.

- The **universal set** $U$ contains all objects under consideration.

## Subsets

- A set $A$ is a **subset** of a set $B$ if every element of $A$ is also an element of $B$. We write $A \subseteq B$.

- We say that two sets $A$ and $B$ are equal, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

- If $A \subseteq B$ and $A \neq B$, then $A$ is a **proper subset** of $B$, written $A \subsetneq B$.

## Operations on Sets

Let $A$ and $B$ be subsets of a universal set $U$.

- **Union:** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

- **Intersection:** $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

- **Difference:** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

- **Complement:** $A^c = U \setminus A$.

## Venn Diagrams

Venn diagrams are visual representations of sets using regions in the plane. They are useful for understanding unions, intersections, and complements.
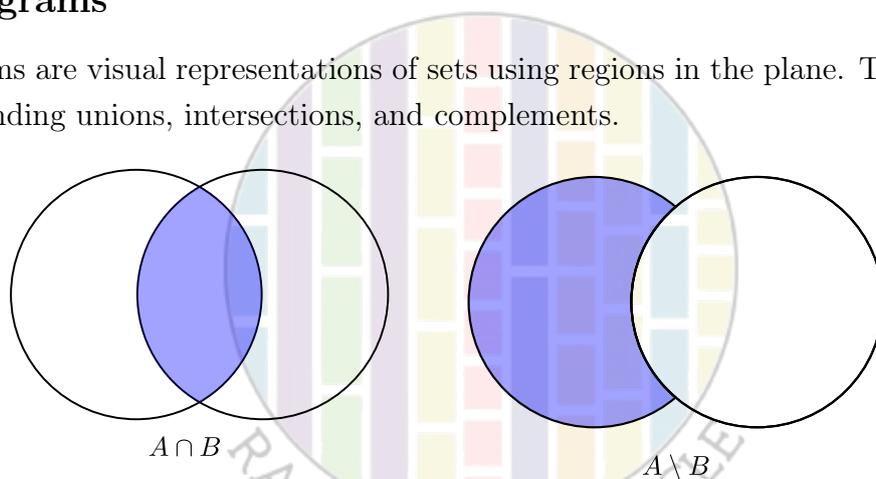


Figure 1: Ven diagram

## Cardinality

The **cardinality** of a finite set $A$, denoted $|A|$, is the number of elements in $A$.

$$A = \{a, b, c\} \Rightarrow |A| = 3$$

## Power Set

The **power set** of $A$, denoted $\mathcal{P}(A)$, is the set of all subsets of $A$.

Some notations:

- $\mathbb{N}$ : the set of natural numbers, $\{1, 2, \ldots, \}$

- $\mathbb{Z}$ : The set of integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

- $\mathbb{Q}$ : The set of rational numbers,

- $\mathbb{R}$ : The set of real numbers,

**Problem 1.** Describe the elements of the following sets.

(i) The set of people with 5 heads.

(ii) $A = \{x \mid x \text{ is a number less than } 5\}$.

(iii) $B = \{n \in \mathbb{N} : n \text{ is an even prime number}\}$.

(iv) $C = \{x \in \mathbb{N} : x^2 = 9\}$.

(v) $D = \{x \in \mathbb{Z} : x^2 = 9\}$.

(vi) $E = \{x \in \mathbb{R} : x < 0\}$.

(vii) $F = \{x \in \mathbb{R} : x(x-1)(x+2) < 0\}$.

**Problem 2.** Show that $\mathbb{N} \subsetneq \mathbb{Z}$. Is it true that $\mathbb{Z} \subseteq \mathbb{N}$?

**Problem 3.** Are the following sets equal?

1. $A = \{h : h \text{ is a human being with 6 legs}\}$, $B = \{x \in \mathbb{Q} : x^2 - 2 = 0\}$.

2. $A = \{x \in \mathbb{Z} : x(x-1)(x-2) = 0\}$, $B = \{x \in \mathbb{Q} : (x^2 - 1) = 0\} \cup \{x \in \mathbb{Q} : x^2 = 0\}$.

**Problem 4.** Let

$$A = \{x \in \mathbb{R} \mid x^2 > x + 6\} \quad \text{and} \quad B = \{x \in \mathbb{R} \mid x > 3\}.$$

Determine which of the following statements are true:

1. $A \subseteq B$

2. $B \subseteq A$

**Problem 5.** Show that $\varnothing \subset A$ for every set $A$. Also find:

$$A \cup \varnothing \quad \text{and} \quad A \cap \varnothing.$$

**Problem 6.** Let

$$A = \{\varnothing, \{\varnothing\}\}.$$

Answer the following questions:

  (i) Is $\varnothing \in A$?

  (ii) Is $\{\varnothing\} \in A$?

  (iii) Is $\{\varnothing\} \subset A$?

  (iv) Is $\{\{\varnothing\}\} \in A$?

  (v) Is $\{\{\varnothing\}\} \subset A$?

**Problem 7.**

  (i) If $|A| = 5$, how many elements are in the power set $\mathcal{P}(A)$?

  (ii) Find a set $A$ such that: $A \cap \{1, 2, 3, 4\} = \{2, 4\}$.

  (iii) Prove or disprove: If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

  (iv) The universal set is all integers from 1 to 20. Let $A$ be the multiples of 2 and $B$ the multiples of 3.

    (a) Find $A \cap B$

    (b) How many elements are in $A \cup B$?

**Problem 8.** Let $S$ be a subset of the natural numbers $\mathbb{N}$ such that whenever a positive integer $n$ belongs to $S$, the number $n+1$ also belongs to $S$. Describe all possible sets $S$ with this property.

**Problem 9.** Let $\Omega = \mathbb{N}$,

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6\}.$$

Find the following sets:

$$A \cup B, \quad A \cap B, \quad A \setminus B, \quad B \setminus A, \quad A \triangle B.$$

**Problem 10.** Prove that $A \cap B \subseteq B$. Then show that

$$A \cap B = B \quad \text{if and only if} \quad B \subseteq A.$$

**Problem 11.** What is the complement of the complement of a set? That is, simplify:

$$(A^c)^c.$$

**Problem 12. De Morgan's Law.**

(i) $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$.