# Krea - RAM - Maths Circle - Session 16

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## 1 Overview

This session introduced equivalence relations through a concrete-to-abstract progression: classifying ourselves, countries, numbers, and ultimately points in $\mathbb{R}^2$. Students discovered the fundamental properties of equivalence relations and partitions through guided exploration, culminating in the construction of quotient spaces like the torus.

## 2 Part 1: Classifying Ourselves

**Central question:** "In how many ways can we divide everyone in this room into groups?"
Students proposed various classifications:

- **Gender:** Male, Female, Non-binary
- **Age:** By year (10, 11, 12, 13, . . . )
- **Class/Grade:** 7th, 8th, 9th
- **Height:** $h < 150$cm, $150 \leq h < 170$cm, $h \geq 170$cm
- **First letter of name:** A, B, C, . . . , Z
- **Birth month:** January, February, . . . , December
- **Handedness:** Left, Right, Ambidextrous



### Key Observations

1. Each person belongs to *exactly one* group per classification
2. Different classifications yield different numbers of groups
3. Some classifications are "finer" than others (age vs. decade of birth)
4. Trivial classifications exist: everyone together, everyone separate

### Formalization

If we classify by gender $G$, we partition the room into classes: {Male, Female, NB, ...}.
For person $p$ in group $g$: $p \in g$.
The groups partition the room:

$$\bigcup_{g \in G} g = \text{Everyone}, \quad g_i \cap g_j = \emptyset \text{ for } i \neq j$$

### Problems

(a) List 5 different ways to classify the students in this room.

(b) Can you find a classification that creates exactly 3 groups? Exactly 7?

## 3 Part 2: Classifying Countries

**Activity:** Given a world map, classify countries into groups.
Students proposed:

- **Continent:** Asia, Africa, Europe, North America, South America, Australia, Antarctica
- **Set of colours in the flag:** {Red, White, Blue}, {Green,White}, etc.
- **Population:** $< 10$M, 10M–100M, 100M–1B, $> 1$B
- **Government:** Democracy, Monarchy, Dictatorship, Other
- **Hemisphere:** Northern, Southern, Eastern, Western
- **Climate:** Tropical, Temperate, Arctic, Desert
- **Driving side:** Left, Right

### Critical Discussion

Some classifications are unambiguous (hemisphere by equator), others are fuzzy (climate).
**Edge cases:** What about transcontinental countries?

- Russia spans Europe and Asia
- Egypt spans Africa and Asia (Sinai Peninsula)
- Turkey spans Europe and Asia

**Resolution:** We must make a choice or refine our classification rules.

### Requirements for Valid Classification

1. **Completeness:** Every country belongs to some group
2. **Exclusivity:** No country belongs to multiple groups in the same classification
3. **Non-triviality:** Groups must be non-empty

### Problems

(a) Choose 3 classification criteria and determine how many groups each creates.

(b) Find an example where a country might naturally belong to two groups. How do you resolve this?

# 4 Part 3: Classifying Numbers

## 4.1 Part 3A: Integers

**Question:** "How can we classify all integers?"
Student suggestions:

- Even/Odd
- Positive/Negative/Zero
- Single-digit/Double-digit/...
- Prime/Composite/Special $(0, 1, -1)$
- Divisible by 3 (remainder 0, 1, 2)
- Perfect squares vs. non-squares

### Divisibility Classification

Classify integers by remainder when divided by $n$.
**For** $n = 2$**:** $\{0 \bmod 2, 1 \bmod 2\} = \{\text{even}, \text{odd}\}$
**For** $n = 3$**:** $\{0 \bmod 3, 1 \bmod 3, 2 \bmod 3\}$

$$[0] = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$$
$$[1] = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$$
$$[2] = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$$

**For** $n = 12$**:** $\{0 \bmod 12, 1 \bmod 12, \ldots, 11 \bmod 12\}$
**Key insight:** Number of groups $= n$.

## Notation and Equivalence

We write $a \equiv b \pmod{n}$ if $a$ and $b$ have the same remainder when divided by $n$.

This is an *equivalence relation*:

1. **Reflexive:** $a \equiv a \pmod{n}$ ✓
2. **Symmetric:** $a \equiv b \pmod{n} \implies b \equiv a \pmod{n}$ ✓
3. **Transitive:** $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n} \implies a \equiv c \pmod{n}$ ✓

The equivalence classes are **residue classes modulo** $n$, denoted $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}_n$.

## 4.2 Part 3B: Rationals

**Question:** "Can we classify rational numbers similarly?"

Classifications:

- By sign: Positive, Negative, Zero
- Integer vs. non-integer
- By denominator in lowest terms: $\{\ldots, \frac{p}{2}, \frac{q}{3}, \frac{r}{4}, \ldots\}$
- Terminating vs. repeating decimals
- By distance from 0: $[0, 1)$, $[1, 2)$, $[2, 3)$, ...

## 4.3 Part 3C: Reals

**Question:** "What about all real numbers?"

Classifications:

- By sign
- By integer part: $[\ldots, [-2, -1), [-1, 0), [0, 1), [1, 2), \ldots]$
- By interval: $[0, 1]$, $(1, 2]$, $(2, 3]$, ...
- Algebraic vs. transcendental
- Rational vs. irrational

**Important observation:** Some classifications are "compatible" with arithmetic:

- Sign preserves multiplication structure
- Integer part preserves ordering
- Modular arithmetic mod$n$ preserves addition and multiplication

# 5 Part 4: Classifying $\mathbb{R}^2$

**Main question:** "How can we partition the $xy$-plane into regions?"

## 5.1 Part 4A: Simple Classifications

Students proposed:

- **Quadrants:** $\{x > 0, y > 0\}$, $\{x < 0, y > 0\}$, $\{x < 0, y < 0\}$, $\{x > 0, y < 0\}$ (plus axes)
- **Horizontal strips:** $y \in [0, 1)$, $y \in [1, 2)$, $y \in [2, 3)$, ...
- **Vertical strips:** $x \in [0, 1)$, $x \in [1, 2)$, $x \in [2, 3)$, ...
- **Concentric circles:** $\sqrt{x^2 + y^2} \in [0, 1)$, $[1, 2)$, $[2, 3)$, ...

## 5.2 Part 4B: Equivalence Relations on $\mathbb{R}^2$

**Definition:** Two points $(x_1, y_1)$ and $(x_2, y_2)$ are equivalent if they satisfy relation $\sim$.

### Example 1: Same $x$-coordinate

$(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2$
**Equivalence classes:** Vertical lines $\{(c, y) : y \in \mathbb{R}\}$ for each $c \in \mathbb{R}$

### Example 2: Same $y$-coordinate

$(x_1, y_1) \sim (x_2, y_2) \iff y_1 = y_2$
**Equivalence classes:** Horizontal lines

### Example 3: Same distance from origin

$(x_1, y_1) \sim (x_2, y_2) \iff x_1^2 + y_1^2 = x_2^2 + y_2^2$
**Equivalence classes:** Circles centered at origin $\{(x, y) : x^2 + y^2 = r^2\}$ for $r \geq 0$

### Example 4: Lines of slope 1

$(x_1, y_1) \sim (x_2, y_2) \iff y_1 - x_1 = y_2 - x_2$
**Equivalence classes:** Lines $y = x + c$ for $c \in \mathbb{R}$

### Example 5: Unit squares

$(x_1, y_1) \sim (x_2, y_2) \iff \lfloor x_1 \rfloor = \lfloor x_2 \rfloor$ and $\lfloor y_1 \rfloor = \lfloor y_2 \rfloor$
**Equivalence classes:** Unit squares $[n, n + 1) \times [m, m + 1)$ for $n, m \in \mathbb{Z}$