RAM Maths Circle

December 14, 2025

Nagpur

Introduction

Today's session was a lecture on *Invariants*. The class began with a short demonstration to introduce the idea of a quantity that remains unchanged under specified moves, followed by a sequence of problems chosen to highlight different flavours of invariant reasoning: parity, modular arguments, conservation, and strategy-based invariants. The aims were to develop intuition about what to look for when searching for invariants, and to practise formulating and applying invariant-based arguments in a range of combinatorial settings.

Demonstration used to introduce the concept

Upside-down cups demonstration.

1. Setup: Seven cups are placed upside down on the table.
2. Task given to students: Turn cups so that all are standing right-side up, with the constraint that on each move you must flip exactly two cups (you choose any two).
3. Students were given time to attempt the task and asked to record what changes and what appears to remain unchanged after each move.
4. Instructor prompts guided students to suggest candidate invariant quantities and to test them by performing a few moves.

Problems

Problem 1

A bag contains 99 red marbles and 99 blue marbles. Repeatedly perform the following operation: draw two marbles at random, then

- if the two marbles are the same colour, put a red marble back into the bag;
- if the two marbles are different colours, put a blue marble back into the bag.

Continue until only one marble remains. What is the colour of that marble?

Problem 2

Alice writes the numbers $1, 2, 3, 4, 5, 6$ on a blackboard. Bob repeatedly selects two numbers, erases them, and writes their sum on the board. For example, choosing 3 and 4 replaces them with 7. Bob continues until only one number is left. What values are possible for that final number?

Problem 3

A dragon has 100 heads. A knight can cut off specific numbers of heads with one blow of his sword, but heads grow back immediately according to the following rules:

- Cut off **15** heads $\rightarrow$ **24** heads grow back.
- Cut off **17** heads $\rightarrow$ **2** heads grow back.
- Cut off **20** heads $\rightarrow$ **14** heads grow back.
- Cut off **5** heads $\rightarrow$ **17** heads grow back.

The dragon dies if and only if all heads are blown off (0 heads remain). **Question:** Can the dragon ever die?

Problem 4

A magic field initially has 1000 flowers. Visitors may pick flowers, but doing so causes new ones to grow. The rules are:

- Pick **30** flowers $\rightarrow$ **102** grow back.
- Pick **90** flowers $\rightarrow$ **144** grow back.
- Pick **102** flowers $\rightarrow$ **12** grow back.
- Pick **120** flowers $\rightarrow$ **84** grow back.

The magic field will close permanently if and only if the field is left with exactly 0 flowers. **Question:** Will the magic field ever close?

Problem 5

There are 13 black, 15 blue and 17 red chameleons on an island. When two chameleons of different colours meet, they both change their colour to the third colour. Is it possible that after some time all the chameleons on the island are the same colour?

The Take-Away Stones Battle (NIM Style)

Setup:

- A pile of 21 stones is placed on the table.
- Two teams take turns removing 1, 2, or 3 stones.
- The team that takes the last stone wins.

Twist: Team A tries to win, while Team B's goal is merely to block Team A from winning. Students were asked to find a strategy that guarantees a win. And to find some invariant here.

Exploration

- Students tested small cases of the marble, chameleon, and flower problems to observe patterns and propose possible invariant quantities such as parity or values modulo a fixed number.
- Groups discussed how each allowed move alters the system and which combinations of counts or sums remain unaffected.
- The problems encouraged students to distinguish between what changes and what must remain constant, leading to discussions about impossibility and necessary conditions.
- A short take-away stones game was used to illustrate how invariants can also guide winning strategies in sequential games.