RAM Math Circle - Chennai

Session Report - November 11 2024

1 Magic Squares

Let n be a positive integer.

Definition: A $n \times n$ magic square is a grid with n rows and n columns filled with number so that the sums of each of the rows are equal, and are equal to the sums of each of the columns and sums of each of the two diagonals. This common sum is sometimes called the *magic sum* of the given magic square.

For example, the following grid of numbers is a 3×3 magic square because the sum of each row, each column and each diagonal equals 15.

4	3	8
9	5	1
2	7	6

Consider magic squares that satisfy the following restrictions:

- The numbers to be filled should be from the given set of numbers, unless otherwise specified.
- No number should be repeated in filling the magic square.

We attempted to solve the following question about 3×3 magic squares:

Question: How many different 3×3 magic squares can be obtained if the numbers to be filled are from the set $\{1, 2, 3, \ldots, 9\}$?

Sketch of proof: First we note that if the filling is to be from the set $\{1, 2, 3, \ldots, 9\}$, then the magic sum must be 15. This is because, if the magic sum is X, then we get the equation:

$$\frac{1 + 2 + 3 + \cdots + 9}{3} = X.$$

The LHS evaluates to $\frac{90}{2 \times 3} = 15$. So the magic sum must be 15.

Suppose the 3×3 square is filled as follows:

a	b	c
d	e	f
g	h	i

Then we get the following equations:

a+b+c=15	a+d+g = 15	a+e+i = 15
d+e+f = 15	b+e+h = 15	c+e+g=15
g+h+i = 15	c+f+i = 15	

Using the above equations, you can prove that

1. The number appearing in the center must be 5, that is $e = 5$.
2. The number 1 cannot appear a a corner position.

(Please try to prove the above statements).

Using these statements you can argue further to conclude that a possible filling is

6	1	8
7	5	3
2	9	4

and all other fillings are obtained from either rotations of reflections of the given filling.

(Some students were able to solve this problem, and some students ventured into other problems while thinking about this one. We have listed some of the other problems for students to try and solve.)

2 Exercises to try

1. Suppose you are given a $n \times n$ magic square filled with numbers from the set $\{1, 2, \ldots, n^2\}$. What is the magic sum? You can try solving this problem for $n = 3, 4, 5$ first.

2. Fill a 3×3 magic square with the numbers $11, 12, 13, 21, 22, 23, 31, 32$ and 33.

3. Some students were familiar with the so-called 'knight's move' strategy for filling up the magic square with any given set of numbers. We haven't discussed this strategy on class, however here is a question that can be asked w.r.t this strategy:

   "Does this strategy work for any given set of numbers or only for number that satisfy certain properties?"

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