

# Krea - RAM - Maths Circle - Session 14

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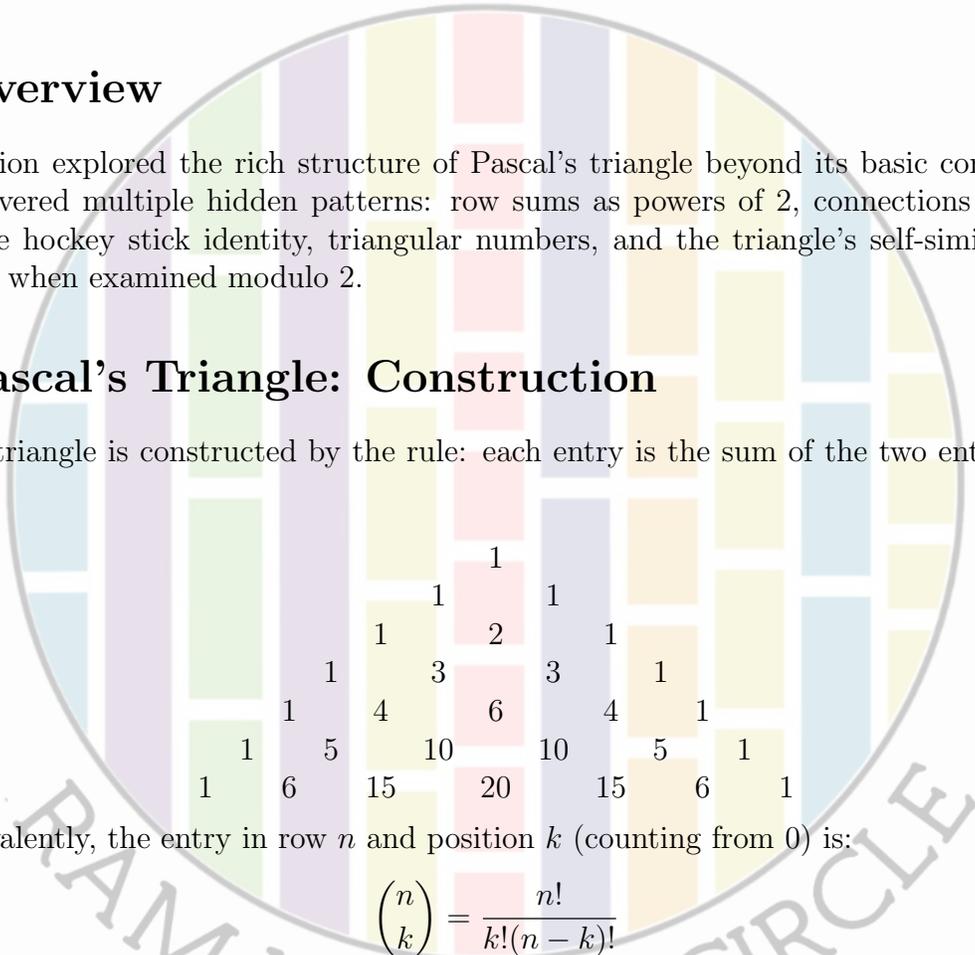
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## 1 Overview

This session explored the rich structure of Pascal's triangle beyond its basic construction. We discovered multiple hidden patterns: row sums as powers of 2, connections to powers of 11, the hockey stick identity, triangular numbers, and the triangle's self-similar fractal structure when examined modulo 2.

## 2 Pascal's Triangle: Construction

Pascal's triangle is constructed by the rule: each entry is the sum of the two entries above it.



				1					
			1	2	1				
		1	3	6	10	15			
	1	4	10	20	35	56	70		
1	6	15	35	70	126	210	280	252	182

Equivalently, the entry in row  $n$  and position  $k$  (counting from 0) is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

## 3 Pattern 1: Row Sums

**Discovery:** Add all entries in each row:

Row 0:  $1 = 1 = 2^0$

Row 1:  $1 + 1 = 2 = 2^1$

Row 2:  $1 + 2 + 1 = 4 = 2^2$

Row 3:  $1 + 3 + 3 + 1 = 8 = 2^3$

Row 4:  $1 + 4 + 6 + 4 + 1 = 16 = 2^4$



## 6 Pattern 4: Triangular Numbers



**Discovery:** The third diagonal contains the triangular numbers: 1, 3, 6, 10, 15, 21, ...

These are the numbers of dots needed to form triangular patterns:

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

Similarly:

- **Fourth diagonal:** Tetrahedral numbers  $\binom{n}{3} = 1, 4, 10, 20, 35, \dots$
- **Fifth diagonal:** Pentatope numbers  $\binom{n}{4} = 1, 5, 15, 35, 70, \dots$

## 7 Pattern 5: Sierpiński's Triangle

**Discovery:** Color each odd entry in Pascal's triangle black and each even entry white. A fractal pattern emerges!

For row  $n = 2^k - 1$  (rows 1, 3, 7, 15, 31, ...), the pattern exhibits perfect self-similarity.

**Mathematical insight:** Entry  $\binom{n}{k}$  is odd if and only if there is no carry when adding  $k$  and  $n - k$  in binary.

By Lucas's theorem:  $\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \pmod{2}$ , where  $n_i, k_i$  are the  $i$ -th bits of  $n$  and  $k$  in binary.

### Problems

- Which entries in row 7 are odd? Verify using the binary addition rule.
- Draw the first 16 rows of Pascal's triangle and shade odd entries. What fractal pattern appears?
- Prove that if  $n = 2^k$ , then row  $n$  has exactly two odd entries:  $\binom{n}{0}$  and  $\binom{n}{n}$ .

## 8 Pattern 6: Binomial Theorem Connection

Pascal's triangle encodes the binomial expansion:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Row 3 gives:  $(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

Setting  $x = y = 1$  recovers the row sum formula  $2^n$ . Setting  $x = 10, y = 1$  gives the powers of 11.

## Problems



- (a) Expand  $(x + y)^5$  using row 5 of Pascal's triangle.
  - (b) What does  $(1 - 1)^n$  tell us about alternating sums of rows?
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## 9 Take-Home Challenges

1. Find the sum of squares along each row:  $\sum_{k=0}^n \binom{n}{k}^2$ . What pattern emerges?
2. Investigate what happens when you color entries by their remainder when divided by 3 or 5.
3. Find other patterns in the Pascal's Triangle.

