RAM Math Circle - Chennai

# Synopsis for January 11 2026

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We continued the math session with separate classes for the junior and senior batches. The junior batch carried on with number theory from the previous session, while the senior batch did geometry along with problems from previous week.

## Junior Batch

We started the session by briefly recalling the following definitions:

- **Prime numbers:** A natural number greater than 1 is called a prime number if it has exactly two positive divisors, namely 1 and itself.
- **Divisor of a number:** A natural number $d$ is called a divisor of a natural number $n$ if $d$ divides $n$ exactly, that is, $n = dk$ for some natural number $k$.

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We then started by listing the divisors of 10, 12, and 360. From these examples, we quickly observed that divisors usually occur in pairs. Based on this observation, some students suggested that the number of divisors of a number is always even. This claim was quickly questioned when many students pointed out that 4 has exactly 3 divisors, namely $1, 2$, and 4. Similarly, 16 has 5 divisors.

This led to the conjecture that the number of divisors of a number is odd if and only if the number is a perfect square. If $n$ is not a perfect square, then each divisor $d$ pairs with a distinct divisor $\frac{n}{d}$, and these two are different. Hence, the divisors come in pairs, giving an even number of divisors.

If $n$ is a perfect square, say $n = m^2$, then the divisor $m$ pairs with itself. All other divisors come in pairs, and this single unpaired divisor makes the total number of divisors odd.

Next, we listed the divisors of $7^8$. They are 1, 7, $7^2$, $7^3$, $7^4$, $7^5$, $7^6$, $7^7$, $7^8$. We realised that the fact that 7 is a prime number is not important here. For any prime number $p$, the divisors of $p^8$ are 1, $p$, $p^2$, $p^3$, $p^4$, $p^5$, $p^6$, $p^7$, $p^8$.In general, if $p$ is a prime number and $k$ is a natural number, then $p^k$ has exactly $k + 1$ divisors, namely 1, $p$, $p^2$, $\ldots$, $p^k$.

Next, we moved to numbers which are products of exactly two different prime numbers. Such numbers are of the form $n = p^a \times q^b$, where $p$ and $q$ are prime numbers with $p \neq q$, and $a$ and $b$ are natural numbers.

If $a = 3$ and $b = 5$, then $n = p \times p \times p \times q \times q \times q \times q \times q$. Any divisor of $n$ must be of the form $p^x \times q^y$, where

$$0 \leq x \leq 3 \quad \text{and} \quad 0 \leq y \leq 5.$$

There are 4 choices for $x$ and 6 choices for $y$, so the total number of divisors is $4 \times 6 = 24$.

We observed that the exact values of $p$ and $q$ are not important. In general, the number $n = p^a \times q^b$ has exactly $(a + 1)(b + 1)$ divisors.

After some discussion, we arrived at the general result:

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then the number of divisors of $n$ is

$$(a_1+1)(a_2+1)\cdots(a_r+1).$$

Next, we studied the **sum of divisors** of a natural number. We first computed:

- Sum of divisors of $10 = 1+2+5+10 = 18$.
- Sum of divisors of $12 = 1+2+3+4+6+12 = 28$.
- Sum of divisors of $64 = 1+2+4+8+16+32+64 = 127$.

We quickly realised that listing all divisors is not a convenient method for large numbers such as 360. So, we again started with numbers of the form $p^k$, where $p$ is a prime number and $k$ is a natural number.

The divisors of $p^k$ are $1, p, p^2, \ldots, p^k$.

Let

$$S = 1 + p + p^2 + \cdots + p^k. \tag{1}$$

Multiplying both sides by $p$, we get

$$pS = p + p^2 + \cdots + p^k + p^{k+1}. \tag{2}$$

Subtracting the first equation from the second,

$$pS - S = p^{k+1} - 1, \tag{3}$$
$$S(p-1) = p^{k+1} - 1. \tag{4}$$

Hence,

$$S = \frac{p^{k+1}-1}{p-1}.$$

Next, we found the sum of divisors of $p^3 \times q^5$, where $p$ and $q$ are prime numbers and $p \neq q$. We listed the divisors in the following table:

| | | | | | |
|---|---|---|---|---|---|
| $1$ | $q$ | $q^2$ | $q^3$ | $q^4$ | $q^5$ |
| $p$ | $pq$ | $pq^2$ | $pq^3$ | $pq^4$ | $pq^5$ |
| $p^2$ | $p^2q$ | $p^2q^2$ | $p^2q^3$ | $p^2q^4$ | $p^2q^5$ |
| $p^3$ | $p^3q$ | $p^3q^2$ | $p^3q^3$ | $p^3q^4$ | $p^3q^5$ |

Each row is a multiple of

$$1 + q + q^2 + q^3 + q^4 + q^5 = \frac{q^6-1}{q-1}.$$

Adding all rows together, we get

$$S = (1+p+p^2+p^3)\left(\frac{q^6-1}{q-1}\right) \tag{5}$$
$$= \left(\frac{p^4-1}{p-1}\right)\left(\frac{q^6-1}{q-1}\right). \tag{6}$$

Students observed that the exact exponents are not important. In general,

$$\text{Sum of divisors of } p^a \times q^b = \left(\frac{p^{a+1}-1}{p-1}\right)\left(\frac{q^{b+1}-1}{q-1}\right).$$

Some students further claimed that if

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

then the sum of divisors of $n$ is

$$\left(\frac{p_1^{a_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{a_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_r^{a_r+1}-1}{p_r-1}\right).$$

We left it to the students to convince themselves of this result, as some students needed more time to fully appreciate the formula.

After the break, due to lack of time, we moved to an origami activity. Students made a tulip flower using rectangular paper, and the activity was enjoyable and relaxing for the class.

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## Senior Batch

In the previous session we had set up some results required to get to the Euler line. The students were eager to see what lies ahead.

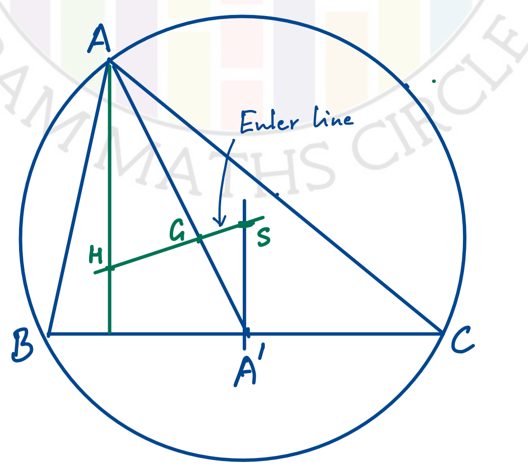
We started by proving the following:

1. Consider the circumcircle of $\Delta ABC$, and consider the chord $CX$ perpendicular to $BC$. Then $CX = AH$, where $H$ is the orthocenter of $\Delta ABC$.

2. Let $A'$ be the midpoint of side $BC$, $H$ be the orthocenter and $S$ be the circumcenter of $\Delta ABC$. Then $AH = 2SA'$.

Finally we could prove:

**Theorem:** In any triangle, the orthocenter $H$, the circumcenter $S$ and the centroid $G$ are collinear. The line passing through these points is called the *Euler line* of the triangle. Moreover, $SG = 2GH$.




We spent some time understanding the import of this result. We had a general discussion on why properties such as collinearity and concurrence are special. There was a general feeling of surprise and awe when we proved the above theorem, and appreciation for all the layers of the geometry of a triangle and its cicumcircle that we ad peeled off on the way to reach this gem.

After the break we recalled all the things we learned so far. THen we discussed another gem: the nine-point circle of a triangle. We talked about strategies we could use to prove the existence of the nine-point circle.