IIIT Delhi - RAM Maths Circle
Session 10

(Organized by the Department of Mathematics, IIIT Delhi)

IIIT-Delhi November 16th, 2025

Please attempt the problems with * once you are done with the other problems.

1 Some New Definitions

Problem 1. In a country, there are 15 towns, each connected to at least 7 others. Prove that one can travel from any town to any other, possibly passing through some towns in between.

Definitions.

- A graph is *connected* if any two of its vertices can be connected by a path (a sequence of edges, each beginning where the previous ends).
- A *cycle* is a closed path whose starting and ending vertices coincide.

Problem 2. Prove that a graph with n vertices, each of degree at least $(n-1)/2$, is connected.

A non-connected graph consists of several "pieces," within each of which one can travel along edges from any vertex to any other. These pieces are called the *connected components* of the graph. A connected graph has exactly one connected component.

Problem 3. In Wonderland, there is only one means of transportation: magic carpet. Twenty-one carpet lines serve the capital, Delhi. A single line flies to Shimla, and every other city is served by exactly 20 carpet lines. Show that it is possible to travel from Delhi to Shimla (perhaps with transfers).

Problem 4. In a certain country, 100 roads lead out of each city, and one can travel between any two cities. One road is closed for repairs. Prove that one can still get from any city to any other.

2 Eulerian Graphs

Problem 5. Can one draw the graph pictured in (a) Figure 1; (b) Figure 2, without lifting the pencil from the paper and tracing each edge exactly once?

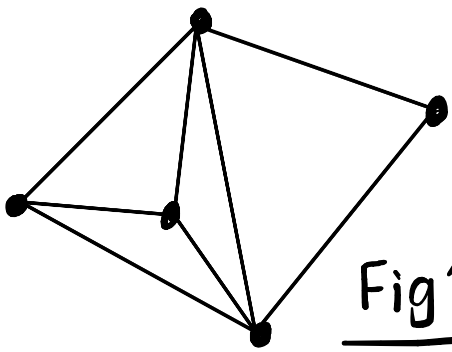


Fig1

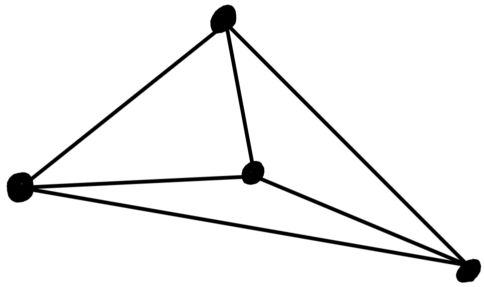
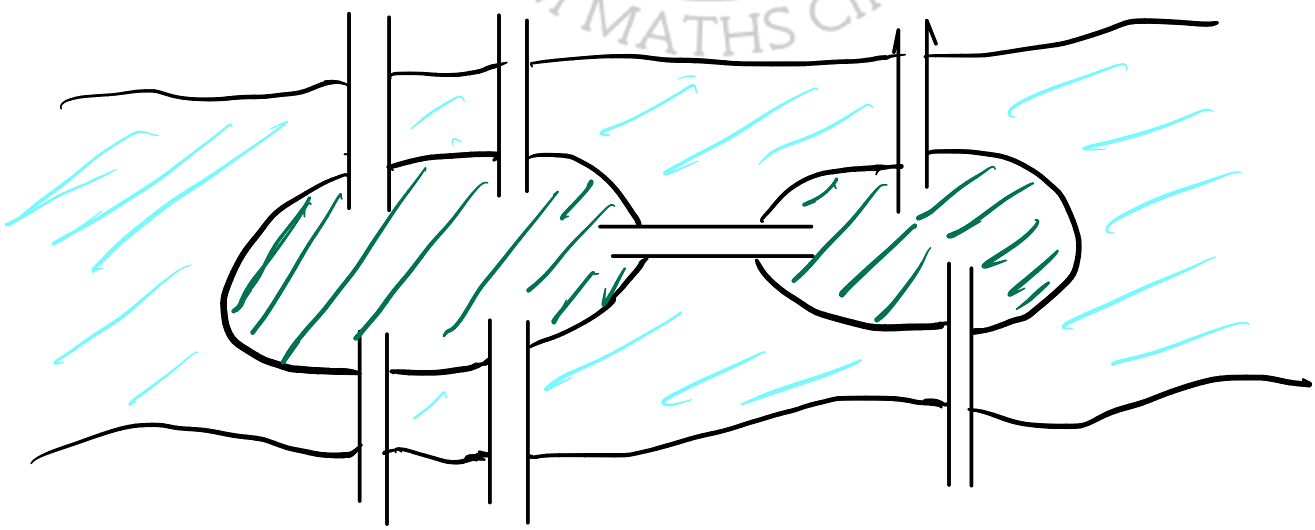


Fig2

A graph that can be traversed without lifting the pencil while tracing each edge exactly once can have *no more than two odd vertices.*

This kind of graph was first studied by Leonhard Euler in 1736 in connection with the famous problem of the Königsberg bridges. Graphs that can be traversed in this way are called *Eulerian graphs.*

Problem 6. A map of the city of Königsberg is given in Figure 38. The city lies on both banks of a river and contains two islands connected by seven bridges. Can one stroll around the town, crossing each bridge exactly once?



Problem 7. A group of islands are connected by bridges in such a way that one can walk from any island to any other. A tourist walked around every island, crossing each bridge exactly once. He visited the island of Thrice three times. How many bridges are there to

Thrice if (a) the tourist neither started nor ended on Thrice; (b) he started on Thrice but didn't end there; (c) he started and ended on Thrice?

Problem 8. (a) A piece of wire is 120 cm long. Can one use it to form the edges of a cube, each edge 10 cm? (b) What is the smallest number of cuts needed in the wire to form the cube?