RAM Math Circle - Chennai

Synopsis for December 21 2025

We continued our math session by conducting classes in two separate classrooms for the junior and senior batches. The junior batch continued their investigations in number theory from the previous session, whereas the senior batch explored topics in geometry.

Junior Batch

The session began with a warm-up activity, where we revised two important methods for finding the HCF of two numbers: the prime factorisation method and Euclid's algorithm. To recall the execution of Euclid's algorithm, we worked through a concrete example by finding the HCF of 2520 and 1980. Using Euclid's algorithm, we carried out the following steps.

$$\begin{aligned} 2520 &= 1980 \times 1 + 540, \\ 1980 &= 540 \times 3 + 360, \\ 540 &= 360 \times 1 + 180, \\ 360 &= 180 \times 2 + 0. \end{aligned}$$

Since the remainder is zero in the final step, we conclude that $\text{HCF}(2520, 1980) = 180$. After completing the computation, we revisited the question of why Euclid's algorithm works and discussed the underlying ideas behind it.

The following assertions were discussed with the students:

- If $m \mid N_1$ and $m \mid N_2$, then does $m \mid (N_1 + N_2)$?
- If $m \mid N_1$ and $m \mid (N_1 + N_2)$, then does $m \mid N_2$?

These observations helped students understand how divisibility behaves under addition and subtraction. We then wrote down the general form:

$$\text{Dividend } (D) = \text{Divisor } (d) \times \text{Quotient } (q) + \text{Remainder } (r).$$

Based on this, we made the key assertion that any common factor of the dividend D and the divisor d must also be a factor of the remainder r.

Proof of the Key Assertion:

Let m be a common factor of D and d. Then

$$m \mid D \quad \text{and} \quad m \mid d.$$

Hence, there exist integers a and b such that

$$D = ma \quad \text{and} \quad d = mb.$$

Using the division algorithm,

$$D = dq + r.$$

Substituting for D and d, we get

$$ma = (mb)q + r.$$

Rearranging,

$$r = ma - mbq = m(a - bq).$$

Since $a - bq$ is an integer, it follows that $m \mid r$.

Thus, any common factor of D and d is also a factor of the remainder r. Therefore, the HCF of D and d is the same as the HCF of d and r. Euclid's algorithm works based on this idea. We revisited the example of 2520 and 1980 and justified each step of Euclid's algorithm using the above principle:

Division Step	**Justification**
$2520 = 1980 \cdot 1 + 540$	Any common factor of 2520 and 1980 divides 540. Hence, $\text{HCF}(2520, 1980) = \text{HCF}(1980, 540)$.
$1980 = 540 \cdot 3 + 360$	Any common factor of 1980 and 540 divides 360. Hence, $\text{HCF}(1980, 540) = \text{HCF}(540, 360)$.
$540 = 360 \cdot 1 + 180$	Any common factor of 540 and 360 divides 180. Hence, $\text{HCF}(540, 360) = \text{HCF}(360, 180)$.
$360 = 180 \cdot 2 + 0$	The remainder is zero, so the divisor 180 is the HCF.

We then considered the following problem:

$$\gcd(2520, 1980) = 180 = 2520 \times (\quad) + 1980 \times (\quad).$$

Students were asked to find integers that fill in the blanks. Using trial and error, they found that

$$180 = 2520 \times (-4) + 1980 \times 5.$$

This led to a more general discussion. For a given pair of natural numbers a and b, consider all integers of the form

$$ax + by,$$

where x and y range over all integers.

Among all positive integers that can be expressed in this form, the smallest one is $\gcd(a, b)$. The session concluded with an open-ended question posed to the students:

Why does the greatest common divisor have this property?

This question was left for further thought and will be explored in more detail in a future session. Towards the end of the session, we discussed an important result relating the HCF and LCM of two natural numbers. Using prime factorisation, we observed that for any two natural numbers a and b,

$$a \times b = \text{HCF}(a, b) \times \text{LCM}(a, b).$$

The result was explained by writing both numbers as products of primes and comparing the powers appearing in the HCF and the LCM. To apply this result, students worked on the problem of finding all pairs of numbers whose HCF is 6 and LCM is 180. We first used the HCF to parametrize the numbers by writing

$$a = 6x, \quad b = 6y,$$

with the condition that $\gcd(x, y) = 1$. Using the identity linking HCF and LCM, this led to the equation

$$xy = 30.$$

Students then listed factor pairs of 30 and checked which pairs were coprime.

x, y	a, b
$1, 30$	$6, 180$
$2, 15$	$12, 90$
$3, 10$	$18, 60$
$5, 6$	$30, 36$

This method allowed them to find all valid pairs of numbers satisfying the given conditions. The session concluded with a homework problem for further practice.

Qn 1. Find all pairs of numbers whose HCF is 12 and LCM is 2520.

Senior Batch

In this session we set out on the path to understanding the Euler line. For this, we revised the topics of concurrency of medians, of perpendicular bisectors, and of altitudes. We also recalled some properties of these entities, and met Fagnano's theorem on the way.

To be specific, we recalled (and in some cases re-proved) following results:

1. The medians of a triangle are concurrent.
2. Properties of the median:
   (a) Each median divides the triangle into two triangles of equal area.
   (b) The 3 medians divide the triangle into 6 smaller triangles of equal area.
   (c) The centroid divides each median in the ratio $2:1$.
3. The altitudes of a triangle are concurrent.
4. Properties of the altitudes and the orthocenter H:
   (a) The vertices of a triangle and the orthocenter form an *orthocentric system* in which each of the four points is the orthocenter of the triangle formed by the remaining three.
   (b) If the altitude AD of triangle ABC meets the circumcircle again at D', then D is the midpoint of HD'.
   (c) In triangle ABC, let H be the orthocenter. The circumcircles of the following triangles have equal radius: ΔABC, ΔAHB, ΔBHC and ΔAHC.

Then we briefly discussed:

- **The orthic triangle:** If D, E and F are the feet of the altitudes of a triangle ABC, then ΔDEF is called the *orthic triangle* of ΔABC.
- **Fagnano's theorem:** The orthic triangle is the triangle of least perimeter that can be inscribed within a triangle.

  Here is a nice video demontrating the proof using reflections and the properties of light.