Session Report - December 15 2024

1 Euler characteristic

Given a graph G, let V denote the number of vertices in G, E denote the number of edges in G and F denote the number of regions (or faces) that the plane is divided into by the edges of the graph G.

Definition: The *Euler characteristic* of G is defined as the number $V - E + F$.

By experimenting with the graphs in the attached worksheet [1], the students discovered for themselves the following theorem.

Euler's formula: For any connected planar graph G, the Euler characteristic is 2.

One can build up the proof of this theorem inductively as follows.

- If G consists of only 1 vertex and no edges, then there's only 1 region and the formula holds.
- Suppose you add a vertex, then you need one edge to make the graph connected. There's still only one region, and the formula holds. Notice that the additional vertex and the new edge cancel out each other's contribution to the formula.
- If we continue to add vertices and edges without creating a new region (i.e. we have a tree), the the formula holds (for the same reason as above).
- To add a new region, we need to add an edge joining two of the *existing* vertices. In this case the new region and the new edge cancel out each other's contribution to the formula.

1.1 Euler characteristic of a line arrangement

Consider a line arrangement of n lines in a plane. Its Euler characteristic is defined similarly as that for a graph, that is, $V - E + F$ where V is the number of vertices (in this case, intersections), E is the number of edges (finite or infinite) and F is the number of regions formed.

By experimenting with line arrangements in the worksheet, it can be noticed that the Euler characteristic of a line arrangement is always 1. This can be proved by associating a graph to the line arrangement as follows:

- associate a vertex to each region formed by the line arrangement
- whenever two regions share a boundary, assign an edge between the corresponding vertices

Observe that the Euler characteristic of the line arrangement is one less than that of the associated connected planar graph (which will always be 2).

Exercise: Try to write a proof of the fact that the Euler characteristic of a line arrangement in a plane is always 1. (Hint: try to prove this inductively as we did for a graph.)

[1] The activities for part of this session were adapted from the work of Joel Hamkins `https://jdh.hamkins.org/tag/euler-characteristic/`

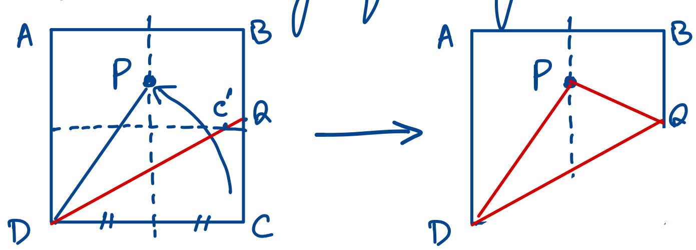
2 Folding angles

We have seen in an earlier session how to fold some common angles at the corner of a square sheet of paper. In this session we discuss how to fold these angles along an edge of a sheet of paper. All these ideas can be consolidated to demonstrate the use of the $30-60-90$ triangle in folding.

2.1 Folding angles at a corner of a square

45° : Obtained by folding a diagonal of the square.

30° :



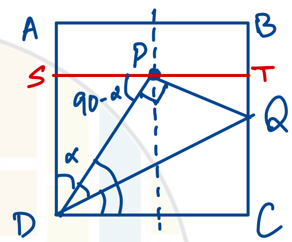
(C maps to P w.r.t. fold DQ.)

Claim: $\angle PDC = 60^\circ$.

In ΔSDP, $l(DP) = l(DC)$

& $l(SP) = \frac{1}{2} l(DC)$

$\therefore l(SP) = \frac{1}{2} l(DP)$.

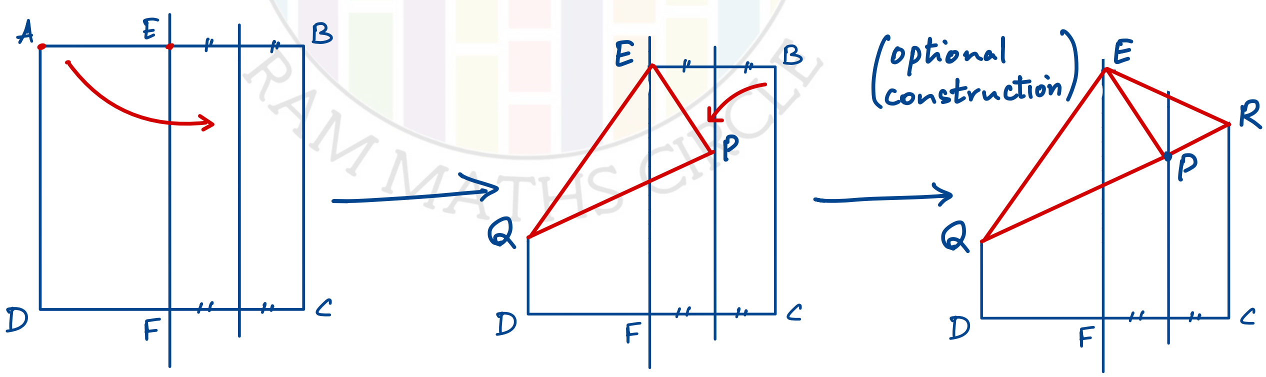


$\therefore \Delta SDP$ is a $30-60-90$ triangle.

$\therefore \alpha = 30^\circ$ & $\angle PDC = 60^\circ$.

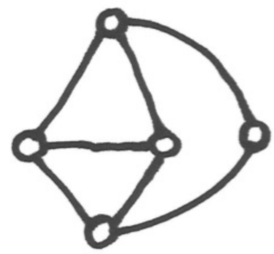
(Can use this to construct 15°).

2.2 Folding angles along an edge



Try to prove that $\angle QEB = 60^\circ$.

A <u>GRAPH</u> IS A COLLECTION OF VERTICES JOINED BY EDGES.



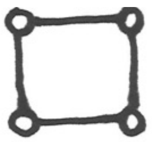
THIS GRAPH HAS

5 VERTICES

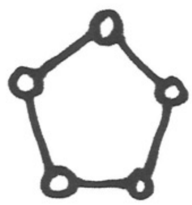
7 EDGES

AND IT DIVIDES THE PLANE INTO

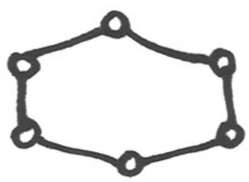
4 REGIONS



V ○ − E ○ + R ○ = ___



V ○ − E ○ + R ○ = ___



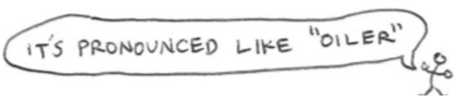
V ○ − E ○ + R ○ = ___

THE MATHEMATICIAN
LEONHARD EULER
NOTICED SOMETHING PECULIAR
WHEN HE CALCULATED:

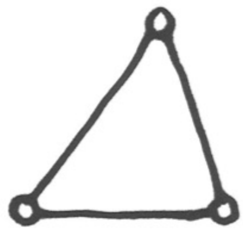
$$\left(\begin{array}{c}\text{NUMBER OF}\\ \textbf{VERTICES}\end{array}\right) - \left(\begin{array}{c}\text{NUMBER OF}\\ \textbf{EDGES}\end{array}\right) + \left(\begin{array}{c}\text{NUMBER OF}\\ \textbf{REGIONS}\end{array}\right)$$

$$V - E + R$$

THIS NUMBER IS
NOW KNOWN AS THE
EULER CHARACTERISTIC.



LET'S CALCULATE IT!



VERTICES ____

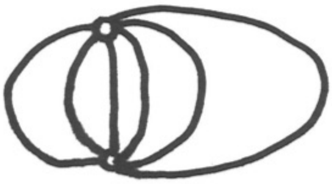
EDGES ____

REGIONS ____

V E R

◯ − ◯ + ◯ = ____

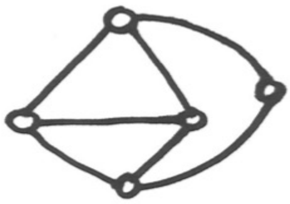
DO WE ALWAYS GET 2?
LET'S TRY SOME EXTREME CASES:



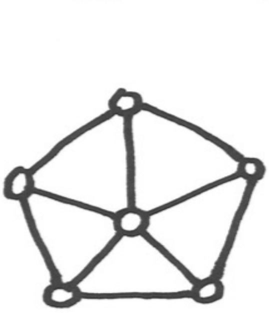
V − E + R = ____

V − E + R = ____

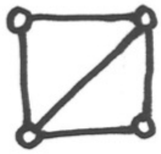
V − E + R = ____



V − E + R = ____

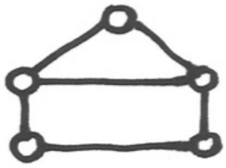


V − E + R = ____

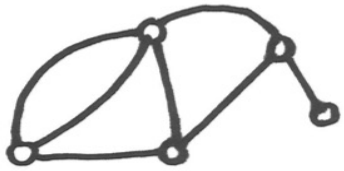


V − E + R = ____

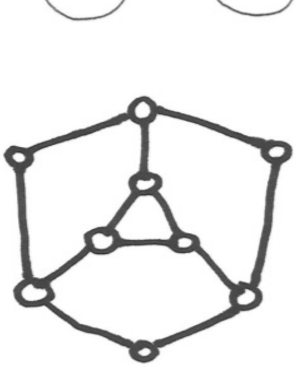
V − E + R = ____



V − E + R = ____

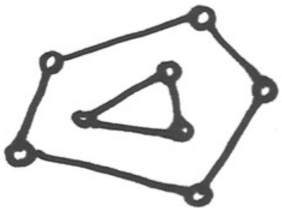


V − E + R = ____



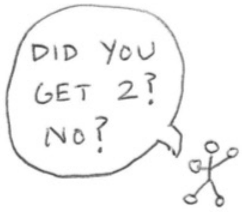
V − E + R = ____

LET'S TRY THIS GRAPH:

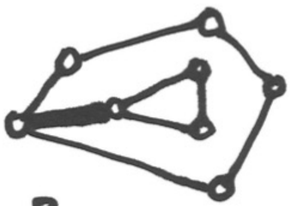


V ◯ − E ◯ + R ◯ = ____

THIS GRAPH IS NOT CONNECTED.



WE CAN CONNECT THE TWO COMPONENTS BY ADDING AN EDGE.



V ◯ − E ◯ + R ◯ = ____

NOW IT WORKS!

TRY IT YOURSELF!

MY GRAPH:

V ◯ − E ◯ + R ◯ = ____

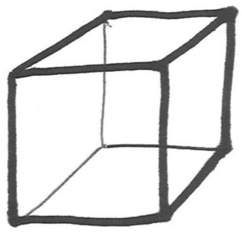
CONCLUSION:

EVERY CONNECTED PLANAR GRAPH HAS

$$V - E + R = 2.$$

LET'S CONSIDER THE SURFACES OF SOME THREE-DIMENSIONAL SOLIDS.

CUBE



VERTICES EDGES FACES

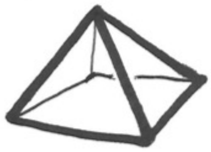
$\bigcirc - \bigcirc + \bigcirc = \_\_\_\_$

WE CALL THEM "FACES" INSTEAD OF "REGIONS".

TRIANGULAR PRISM

$$\overset{V}{\bigcirc} - \overset{E}{\bigcirc} + \overset{F}{\bigcirc} = \_\_\_\_$$

PYRAMID (SQUARE BASE)

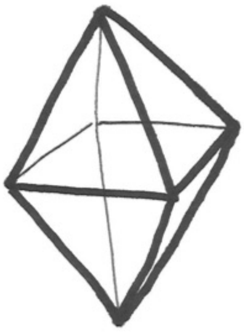


$$\overset{V}{\bigcirc} - \overset{E}{\bigcirc} + \overset{F}{\bigcirc} = \_\_\_\_$$

TETRAHEDRON

$$\overset{V}{\bigcirc} - \overset{E}{\bigcirc} + \overset{F}{\bigcirc} = \_\_\_\_$$

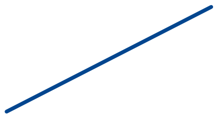
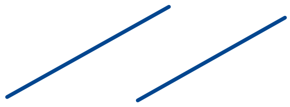
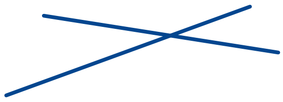
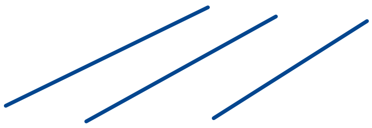
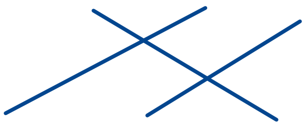
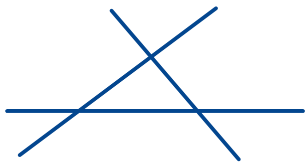
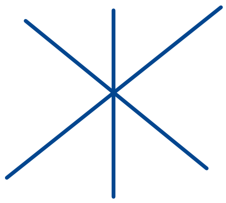
OCTAHEDRON



$$\overset{V}{\bigcirc} - \overset{E}{\bigcirc} + \overset{F}{\bigcirc} = \_\_\_\_$$

Euler characteristic of line arrangements

n = number of lines.

n	arrangements of n lines	Euler characteristic $V - E + F$
1		$0 - 1 + 2 = 1$
2	 (parallel)	$0 - 2 + 3 = 1$
		$1 - 4 + 4 = 1$
3	 (parallel)	$0 - 3 + 4 = 1$
		$2 - 7 + 6 = 1$
		$3 - 9 + 7 = 1$
		$1 - 6 + 6 = 1$

Exercise: Calculate the Euler characteristic for all line arrangements of 4 lines.