RAM Math Circle - Chennai

Synopsis for December 7 2025

Senior batch

We continued to work on Euler's totient function, picking up from the proofs left incomplete last time. Then we worked on some geometry problems related to circles.

1 Divisibility

Notation: For any positive integer n, the notation $\Phi(n)$ denotes the number of positive integers less than and coprime to n. Last time we proved the following statement:

Suppose $(d,n) = 1$. Then for the set $S = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$, number of numbers in S coprime to n is $\Phi(n)$.

Using the above, let's try to prove the following result:

Result 1.1 *Suppose $(m,n) = 1$. Then $\Phi(mn) = \Phi(m)\Phi(n)$.*

- Let $1 \leq k \leq mn$. Then $(mn,k) = 1$ if and only if $(k,m) = (k,n) = 1$ since $(m,n) = 1$.
- Write the mn numbers from 1 to mn in a grid as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & m \\ m+1 & m+2 & m+3 & m+4 & \ldots & 2m \\ 2m+1 & 2m+2 & 2m+3 & 2m+4 & \ldots & 3m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-1)m+1 & (n-1)m+2 & (n-1)m+3 & (n-1)m+4 & \ldots & nm \end{pmatrix}$$

- If k is any number such that $(k,m) = 1$, then every entry in the k^{th} column is coprime to m.
- Now the k^{th} column has __________ elements which are coprime to n.
- So to choose the entries of the grid which are coprime to both m and n we can first choose the k which is coprime to m and then in the k^{th} column we will find __________ entries coprime to n.

  The chosen entries will now be coprime to both m and n.
- Thus the total number of entries (that is numbers between 1 and mn) coprime to both m and n is ________________.

1.1 Exercises

The Fundamental Theorem of Arithemtic (aka the Prime Factorisation Theorem)
For any integer $n > 1$, there exist distinct primes $p_1, p_2, \ldots, p_k$ such that

$$n = p_1^{t_1} p_2^{t_2} p_3^{t_3} \cdots p_k^{t_k}$$

where $t_1, t_2, \ldots, t_k$ are integers ≥ 0.

1. If $n_1, n_2, \ldots, n_k$ are mutually coprime, that is, any two numbers in this list are coprime to each other, then $\Phi(n_1 n_2 \cdots n_k) = \Phi(n_1)\Phi(n_2)\cdots\Phi(n_k)$.
2. How many numbers are coprime to 99?
3. Create a grid of numbers as in the proof above to compile the list of numbers coprime to 99.
4. Write a formula for $\Phi(n)$ where n is any positive integer.

2 Circles

1. When two circles cut each other, then prove that the line joining their centres bisects the common chord at right angles.
2. A chord PQ of a circle cuts a concentric circle at $P'Q'$. Prove that $PP' = QQ'$.
3. Prove that two chords AB and CD of a circle bisect each other if and only if both of them are diameters.
4. If $PQRS$ is a parallelogram whose vertices lie on a circle then PR and QS are diameters of the circle.
5. Two circles cut each other at A, B. IF PAQ and RBS are parallel straight lines meeting the circles again at P, Q, R, S, then prove that $PQ = RS$.

Junior batch

With new batch of students in math circle, we introduced the definitions of prime and composite numbers and explored how to determine whether a number is prime through logical reasoning and case-by-case analysis. Using 103 as our main example, students tested divisibility by small primes and discussed an important logical observation: if a number n is divisible by a number a, then it must also be divisible by every factor of a; therefore, if n is not divisible by a factor of a, it cannot be divisible by a itself. We provided proof of this fact and used it to reduce unnecessary checks when determining primality. Students also learned that to check whether a number n is prime, it suffices to test divisibility only by prime numbers less than or equal to the smallest integer greater than $\sqrt{n}$. Applying these ideas, we confirmed that 103 is prime, and students were given following practice problems:

(a) How many two-digit prime numbers are there?

(b) How many three-digit prime numbers are there?

(c) Is 237421 a prime number?
