Synopsis for January 04 2026

We continued the math session with separate classes for the junior and senior batches. The junior batch carried on with number theory from the previous session, while the senior batch studied standard multiplicative functions and briefly discussed geometry at the end.

Junior Batch

We began with a quick recap of the previous week's homework.

Question
Find all numbers a and b such that $\gcd(a, b) = 18$ and $\operatorname{lcm}(a, b) = 2520$.

Solution Let $\gcd(a, b) = h$ and $\operatorname{lcm}(a, b) = L$. Define $a/h = x$ and $b/h = y$. Here x and y are relatively prime, that is, $\gcd(x, y) = 1$ because h is the highest common factor of a and b.

We use the identity $\gcd(a, b)\operatorname{lcm}(a, b) = ab$. Substituting, $hL = (xh)(yh) = xyh^2$. Dividing both sides by h gives $xy = L/h$.

Here $h = 18$ and $L = 2520$, so $xy = \dfrac{2520}{18} = 140$. We list all factor pairs of 140 such that x and y are relatively prime.

x	y	$a = 18x$	$b = 18y$
1	140	18	2520
5	28	90	504
4	35	72	630
7	20	126	360

The four possible pairs (a, b) are $(18, 2520)$, $(90, 504)$, $(72, 630)$, and $(126, 360)$.

There are exactly **four pairs** (a, b) satisfying the given conditions.

The key result used above is $\gcd(a, b)\operatorname{lcm}(a, b) = ab$. We now give a proof.

Proof
Let the prime factorizations of a and b be $a = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} p_3^{b_3} q_4^{b_4} \cdots q_n^{b_n}$, where p_1, p_2, p_3 are common prime factors, $p_4, p_5, \ldots, p_k$ occur only in a, and $q_4, q_5, \ldots, q_n$ occur only in b.

The highest common factor is

$$\gcd(a, b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} p_3^{\min(a_3,b_3)}.$$

The lowest common multiple is

$$\operatorname{lcm}(a, b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} p_3^{\max(a_3,b_3)} p_4^{a_4} \cdots p_k^{a_k} q_4^{b_4} \cdots q_n^{b_n}.$$

Multiplying, we get

$$\gcd(a, b)\operatorname{lcm}(a, b) = p_1^{\min(a_1,b_1)+\max(a_1,b_1)} p_2^{\min(a_2,b_2)+\max(a_2,b_2)} p_3^{\min(a_3,b_3)+\max(a_3,b_3)} \cdots q_4^{b_4} \cdots q_n^{b_n}.$$

Using $\min(a_i, b_i) + \max(a_i, b_i) = a_i + b_i$, this becomes

$$= p_1^{a_1+b_1} p_2^{a_2+b_2} p_3^{a_3+b_3} \cdots q_4^{b_4} \cdots q_n^{b_n}.$$

Rearranging, we obtain

$$= \left(p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}\right) \left(p_1^{b_1} p_2^{b_2} p_3^{b_3} q_4^{b_4} \cdots q_n^{b_n}\right) = ab.$$

Hence, $\gcd(a, b)\,\mathrm{lcm}(a, b) = ab$.

If $\gcd(a, b) = \mathrm{lcm}(a, b)$, show that $a = b$.
Proof
For any positive integers a and b, $\gcd(a, b) \leq a \leq \mathrm{lcm}(a, b)$ and $\gcd(a, b) \leq b \leq \mathrm{lcm}(a, b)$.

If $\gcd(a, b) = \mathrm{lcm}(a, b)$, then the only possible value for both a and b is this common number. Hence, $a = b$.

Homework
Find all possible integers a, b, and c such that $\gcd(a, b) = 6$, $\gcd(b, c) = 10$, and $\gcd(c, a) = 14$.

This problem requires careful use of prime factorization and compatibility of common factors. Students should clearly justify their constructions.

Senior Batch

We started the session by briefly recalling the following definitions:

- **Prime numbers:** A natural number greater than 1 is called a prime number if it has exactly two positive divisors, namely 1 and itself.
- **Greatest common divisor (gcd):** For two positive integers a and b, the greatest common divisor $\gcd(a, b)$ is the largest positive integer that divides both a and b.
- **Relatively prime (coprime) numbers:** Two numbers a and b are said to be **relatively prime** or **coprime** if $\gcd(a, b) = 1$.
- **Euler Phi function:** For $n \in \mathbb{N}$, the Euler Phi function $\varphi(n)$ is defined as the number of positive integers less than n that are relatively prime to n.

To compute $\varphi(100)$, we need to count the numbers less than 100 that are relatively prime to 100. A number is relatively prime to 100 if and only if it is relatively prime to both 4 and 25. We arrange the numbers from 1 to 100 in a 4×25 grid and remove the numbers that are not relatively prime to 100.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

In each row, exactly five numbers (coloured blue or purple) are not relatively prime to 25. This follows from the property $\gcd(nx + r, n) = \gcd(r, n)$, which shows that the gcd depends only on the remainder r modulo n.

On the other hand, in each column there are exactly two numbers (coloured blue or uncoloured) that are relatively prime to 4. This can also be explained using the same property. Additionally, we observe that each column contains four numbers such that no two have the same remainder modulo 4: $x, x+25, x+50, x+75$ give remainders $x \pmod 4, x+1 \pmod 4, x+2 \pmod 4$, and $x+3 \pmod 4$, respectively.

Columns that are fully coloured contain numbers that are not relatively prime to either 25 or 4. We are interested in counting the uncoloured numbers. There are exactly two such numbers in each of the 20 relevant columns, hence $\varphi(100) = 40$.

A more general discussion gives $\varphi(ab) = \varphi(a)\varphi(b)$ if $\gcd(a,b) = 1$. If $\gcd(a,b) \neq 1$, then in general $\varphi(ab) \neq \varphi(a)\varphi(b)$; for example, $\varphi(8) \neq \varphi(2)\varphi(4)$.

We computed that $\varphi(p^k) = p^k - p^{k-1}$ for a prime p and a positive integer k, and noted that $\varphi(1) = 1$. These results help in computing $\varphi(n)$ for any natural number n. For instance, since $100 = 2^2 \times 5^2$, we get $\varphi(100) = \varphi(4)\varphi(25) = 2 \times 20 = 40$. In general, if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then

$$\boxed{\varphi(n) = \prod_{i=1}^{r} \left(p_i^{a_i} - p_i^{a_i - 1}\right)}$$

Next, we observed the identity $\sum_{d|n} \varphi(d) = n$ by checking it in small cases. For example, when $n = 10$, the divisors of 10 are $1, 2, 5$, and 10. The corresponding values of the Euler Phi function are shown below.

d	1	2	5	10
$\varphi(d)$	1	1	4	4

Similarly, for $n = 12$, the divisors are $1, 2, 3, 4, 6$, and 12, and we have the following table.

d	1	2	3	4	6	12
$\varphi(d)$	1	1	2	2	2	4

In both cases, we verified that the sum of the values in the second row is equal to n. We checked this identity for all prime powers p^k, where p is a prime, and left the general proof as homework.

Homework: For any positive integer n, prove that $\sum_{d|n} \varphi(d) = n$.

One of the key ideas needed for this homework is a systematic way to describe the divisors of a number. For this purpose, we introduced another arithmetic function called the **number of divisors function**. For any n, we denote the number of divisors of n by $\operatorname{div}(n) = \tau(n)$.

For example, $\operatorname{div}(10) = 4$ and $\operatorname{div}(12) = 6$. We computed that $\operatorname{div}(p^k) = k+1$ for a prime p and a positive integer k. We showed that $\operatorname{div}(ab) = \operatorname{div}(a)\operatorname{div}(b)$ only if $\gcd(a,b) = 1$. This condition is essential; for instance, $\operatorname{div}(8) \neq \operatorname{div}(2)\operatorname{div}(4)$. Thus, if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then

$$\boxed{\operatorname{div}(n) = \tau(n) = \prod_{i=1}^{r} (a_i + 1)}$$

Similarly, we introduced another important arithmetic function called the **sum of divisors function**. For a positive integer n, it is defined by $\sigma(n) = \sum_{d|n} d$, where the sum is taken over all positive divisors of n.

From the definition, it is clear that $\sigma(1) = 1$. If p is a prime number, then the only positive divisors of p are 1 and p, and hence $\sigma(p) = p + 1$. We then computed the value of $\sigma(n)$ for prime powers. If p is a prime and k is a positive integer, then the divisors of p^k are $1, p, p^2, \ldots, p^k$. Therefore,

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$

We also observed that the sum of divisors function is multiplicative, but only under a coprimality condition. More precisely, $\sigma(ab) = \sigma(a)\sigma(b)$ holds only if $\gcd(a, b) = 1$. Using this property, we obtained a general formula for $\sigma(n)$. If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the prime factorisation of n, then

$$\boxed{\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}}$$

Towards the end of the session, we discussed the following:

- **The orthic triangle:** Given a triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C, respectively. The triangle $\triangle DEF$ is called the *orthic triangle* of $\triangle ABC$.
- **Fagnano's theorem:** The orthic triangle is the triangle of least perimeter that can be inscribed inside a given acute triangle.
- The students explained a video that demonstrates the proof of Fagnano's theorem using reflections and the laws of light. This video had been suggested to them in the previous session.

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