RAM Math Circle - Chennai

Synopsis for August 24 2025

Graph theory

We started the discussion by reviewing the definitions of graph theory: graphs, edges, vertices, paths, and the degree of a vertex.

- A **graph** is a pair $G = (V, E)$, where V is a set of vertices and E is a set of edges connecting pairs of vertices.
- A **vertex** (or node) is an element in V.
- An **edge** is a connection between two vertices, represented as an element of E.
- A **path** in a graph is a sequence of vertices where each consecutive pair is connected by an edge.
- A **directed graph** is a graph where each edge has a direction, represented as an ordered pair of vertices.
- An **undirected graph** is a graph in which the edges have no direction, represented as unordered pairs of vertices.
- A **loop** is an edge that connects a vertex to itself.
- **Multiple edges** (or parallel edges) are two or more edges that connect the same pair of vertices.
- The **distance** between two vertices u and v, denoted $d(u, v)$, is the length of the shortest path connecting them. If no path exists, $d(u, v) = \infty$.
- The **neighbors of a vertex** v are the vertices directly connected to v by an edge. The set of neighbors of v is denoted by $N(v)$. It is also the set of all vertices u such that $d(u, v) = 1$.
- The **degree of a vertex** is the number of edges incident to that vertex.
- A **connected component** of a graph is a maximal subgraph in which every pair of vertices is connected by a path.
- A **finite graph** is a graph with a finite number of vertices and edges.
- A **simple graph** is an undirected graph with no loops and no multiple edges; each edge uniquely connects two distinct vertices.

We saw several examples of graphs explaining each definition. Through examples, students were made to observe that the total number of vertices of odd degree in a finite simple graph is even. This result is often referred to as the **Handshaking Theorem**. Explorations with examples led students to come up with an informal proof inspired by the "cup game" played two sessions earlier on August 10, 2025. The key idea is: "Adding an edge will not change the parity of the total number of odd-degree vertices." A sketch of the argument is as follows. Let G be a finite simple graph and let us redraw the graph as follows:

- Begin with an empty graph with only vertices and no edges. Here, the total number of odd-degree vertices is zero.
- Add an edge: now the total number of odd-degree vertices becomes two.

- Add another edge: two possibilities arise.
  * If we add the edge in connection with an earlier edge, then the number of odd-degree vertices remains the same.
  * If we add the edge disjoint from the earlier edge, then the number increases by two.
- If we add another edge, in addition to the previous two cases, there is one more possibility: the edge connects two odd-degree vertices, making the number decrease by two.
- Since only a finite number of edges can be added, by mathematical induction, the total number of vertices with odd degree will always remain even.

The following Problems were assigned based on the Handshaking Theorem. Solutions were to be discussed in the next session.

1. In FarFarAway Land there is only one mode of transportation, by magic carpet. Twenty-five carpet lines serve the Capital. A single carpet line serves Smallville, and every other city is served by exactly 10 carpet lines. Show that it is possible to travel by magic carpet from the Capital to Smallville (perhaps with several transfers).

2. Last week, 8 heads of state, including 3 presidents, 3 prime ministers, and 2 emperors, got together for a conference. According to a reporter, each president shook hands with 6 heads of state, each prime minister—with 4, and each emperor—with 1. Is the reporter correct?

3. Can you draw a graph that has 5 vertices, in which the degrees of these vertices are 4, 4, 4, 2, and 2? Either draw such a graph or explain why it is impossible.

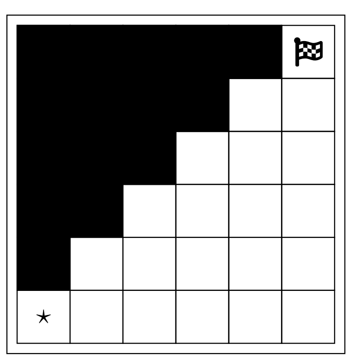
4. This year, 15 students registered for the summer "Hiking with Llamas" backcountry trip. It is known that every participant is acquainted with at least 7 more students from his school who registered for the same trip. Show that all the children are, in fact, from the same school.

5. There are 12 towns on the Island of Knights and Liars; roads connect some of them. Mr. X claims that a different number of roads start in each town. Prove that Mr. X is a Liar.

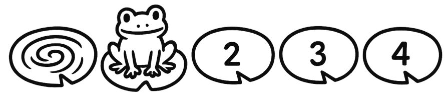
Combinatorics

In the latter half, students explored the following problems related to Catalan numbers. These problems will be a continuation of the path counting theme from the last session, on August 17, 2025. The combinatorics of Catalan numbers will stretch over the next two sessions. Solutions were to be discussed much later, after giving sufficient time in the next session.

Problem 1 In the 6×6 grid of boxes below, start at $\star$ and go to 🏁. You may move only **Right (R)** or **Up (U)** from one box to an adjacent box. You should avoid travelling via black squares. In how many ways you can reach 🏁? Hint: Try for 4×4, 5×5. Can you generalise this problem to 7×7? What can you say for $n \times n$? (remember, all the squares above the diagonals are darkened).



Problem 2 Imagine a little frog sitting on a row of lilly pad(leaf) floating on a pond. The leaves are labeled like this: $\boxed{0}, 1, 2, 3, 4$



- Pad $\boxed{0}$ is dangerous — it's a **whirlpool**! If the frog jumps there, it gets sucked in and disappears forever.
- The frog starts happily on pad 1.
- It can only jump **one pad at a time**, either to the **left** or to the **right**. Jumping from any pad and landing on the pad is **not** allowed. In each jump frog should move to another pad.
- But it must be careful — if it jumps left onto pad 0, it falls into the whirlpool and is gone!

Let's explore how many different ways the frog can jump around without falling into the whirlpool.

> We only count **safe routes** — paths that never land on pad 0 (the whirlpool).

(Qn 1) How many different safe routes can the frog take with 3 jumps to reach pad 3?

(Qn 2) How many different safe routes can the frog take with 10 jumps to reach pad 10?

(Qn 3) How many different safe routes can the frog take with 5 jumps to reach pad 4?

(Qn 4) How many different safe routes can the frog take with 5 jumps to return to pad 1?

(Qn 5) How many different safe routes can the frog take with n jumps to pad 2?

- **(i)** if $k = 2$
- **(ii)** if $k = 3$
- **(iii)** if $k = 4$
- **(iv)** if $k = 5$
- **(v)** what will be your answer, if frog take k jumps?

(Qn 6) Imagine you have "N(very large)" number of lilly pads and If the frog starts at pad 1 and makes k jumps, which pads can it possibly reach? Are there some pads it can *never* land on?

(Qn 7) (**Challenging question**) What connections do you observe between Problem 2 e and Problem 1 which we explored earlier?

More challenging problem:

Let's define: $S(k, n)$ = number of safe routes that take exactly k jumps and end on pad n, starting from pad 1.

> **Important:**
>
> - k = number of **steps** (jumps the frog takes)
> - n = **destination pad** (where the frog ends after those jumps)

RAM MATHS CIRCLE

(**Challenging question**) Find $S(k,n)$.

(**Hint:** for small values k,n say $1 \leq k,n \leq 7$, fill the table of values of $S(k,n)$)

$\downarrow k \backslash n \rightarrow$	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

RAM MATHS CIRCLE