RAM IIITD Delhi Circle - Session 1

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I. Introduction

In this problem set we shall explore the *pigeonhole principle*, and how this seemingly simple observation can lead to the solution of a diverse array of mathematical problems.

Theorem 1 (Pigeonhole Principle)**.** *If $n > k$ and n objects are placed inside k boxes in any arbitrary fashion, then there exists one box with at least two objects.*

Theorem 2 (Generalized Pigeonhole Principle)**.** *If $n > km$ and n objects are placed inside k boxes in any arbitrary fashion, then there exists one box containing at least $m + 1$ objects.*

II. Problems

1) Suppose you place 129 points arbitrarily inside a square of side 8 units. Show that there exists three points which span a triangle of area at most 1.

2) Suppose you place 65 points arbitrarily inside a square of side 1 unit. Show that there exists three points spanning a triangle of area at most $\frac{1}{32}$.

3) Consider 17 points placed arbitrarily inside an equilateral triangle of side 1. Show that there exists a pair of points of distance less than $\frac{1}{4}$.

4) You have 30 boxes arranged in a line and you place 26 balls in these thirty boxes in any arbitrary fashion subject to the condition that no box contains more than one ball. Does there exist 6 consecutive boxes each containing a ball?

5) Consider a collection of 11 positive integers strictly less than 29. Show that there exists a pair of numbers from this collection having a common divisor greater than one.

6) A football team scored 53 goals in total in 30 games, with at least one goal being scored in each game. Show that there was a streak of consecutive games in which a total of six goals were scored (may include a single game of six goals).

7) Consider an 8 by 8 square grid of numbers where 16 entries are 1 and the remaining entries are 0. Show that there exists a pair of ones in the grid such that one of these ones is strictly above and strictly to the right of the other.

8) Consider now a 10 by 10 by 10 cubic grid of numbers with 272 grid entries being 1 and rest being 0. Suppose this grid is in front of you. Show that you will be able to see a pair of ones such that one of these ones is strictly above, strictly to the right, and strictly ahead of the other one.

9) Consider numbers with prime divisors less than 6. Thus, such a number is of the form $2^i3^j5^k$.
   a) We divide such numbers into m different classes, where all numbers inside the same class have same parities (even or odd) for i, j, and k. For example, $2 = 2^1 3^0 5^0$ and $72 = 2^3 3^2 5^0$ belong to the same class, since i for both are odd, and the j and k for both are even. Find the number of such classes m.
   b) Show that if you are given any nine numbers of the above type, the product of two of them must be a perfect square.

10) Consider now a collection of 2003 numbers with prime divisors less than 24.
   a) Show that you can make a collection of 746 distinct pairs of these numbers such that the product of each pair is a perfect square.
   b) Show that the original collection has four numbers whose product is a perfect fourth power.