IIIT Delhi - RAM Maths Circle
Session 3

(Organized by the Department of Mathematics, IIIT Delhi)

IIIT-Delhi September 28th, 2025

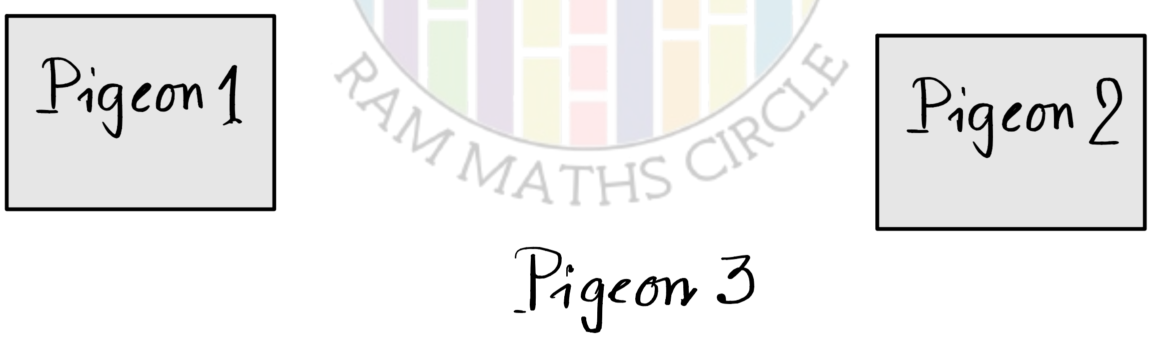
§1. The Pigeon Hole Principle

The well known *Pigeon Hole Principle* is stated as follows:

If we must put $N + 1$ or more pigeons into N pigeon holes, then some pigeon hole must contain two or more pigeons.

You may think it quite obvious! Yes, it is indeed. But how do you prove it?

Notice the vagueness of the proposition “some pigeon hole must contain ...”, “two or more ...”. This is, in fact, a distinguishing feature of the Pigeon Hole Principle, which sometimes allows us to draw quite unexpected conclusions, even when we don't seem to have enough information. (See the following Figure.)



The proof of this principle is quite simple, and uses only a trivial count of the pigeons in their pigeon holes. Suppose you have only one pigeon in each hole. Then there would be no more than N pigeons altogether, which contradicts the assumption that we have $N + 1$ pigeons. This proves the Pigeon Hole Principle, using—and we must be aware of this—the method of proof by contradiction.

Now, you might ask, do the following problems concern pigeons?

1. A bag contains beads of two colours: black and white. What is the smallest number of beads which must be drawn from the bag, without looking, so that among these beads there are two of the same colour?

2. One million pine trees grow in a forest. It is known that no pine tree has more than 600000 pine needles on it. Show that two pine trees in the forest must have the same number of pine needles.

3. Given twelve integers, show that two of them can be chosen whose difference is divisible by 11.

4. The city of Leningrad has five million inhabitants. Show that two of these must have the same number of hairs on their heads, if it is known that no person has more than one million hairs on his or her head.

5. Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates, there are at least nine containing the same sort of apple.

§2. More general pigeons

If you have carefully read the problems above and attempted to solve Problem 5 in the same way as the first two, you may have found it challenging. After all, the standard Pigeonhole Principle only guarantees that there are two crates containing the same type of apples. To tackle this problem, we can instead use the *General Pigeon Hole Principle*:

If we must put $Nk+1$ or more pigeons into N pigeon holes, then some pigeon hole must contain at least $k+1$ pigeons.

In the case $k = 1$, the General Pigeon Hole Principle reduces to the simple Pigeon Hole Principle. We leave the proof of the General Principle as an exercise.

1. In the country of Courland there are M football teams, each of which has 11 players. All the players are gathered at an airport for a trip to another country for an important game, but they are travelling on "standby". There are 10 flights to their destination, and it turns out that each flight has room for exactly M players. One football player will take his own helicopter to the game, rather than travelling standby on a plane. Show that at least one whole team will be sure to get to the important game.

2. Given 8 different natural numbers, none greater than 15, show that at least three pairs of them have the same positive difference (the numbers in any pairs need not be distinct.)

3. Show that in any group of five people, there are two who have an identical number of friends within the group.

4. Several football teams enter a tournament in which each team plays every other team exactly once. Show that at any moment during the tournament, there will be two teams which have played, up to that moment, an identical number of games.

§4. Another generalization

Notice that an arithmetic way of viewing the Pigeon Hole principle is as follows:

If the sum of n or more numbers is equal to S, then among these there must be one or more numbers not greater than S/n, and also one or more numbers not less than S/n.

The reasoning is straightforward and indirect, just like many Pigeonhole arguments. For example, suppose every number were greater than $\frac{S}{n}$. Then their sum would be more than S, which is impossible. Similarly, if every number were smaller than $\frac{S}{n}$, the sum would fall short of S, which again contradicts the assumption. So, there must be at least one number smaller than $\frac{S}{n}$ and at least one number bigger than $\frac{S}{n}$.
We now try to solve a few more problems using this version of the pigeon hole principle.

1. Five young workers received as wages 1500 rubles altogether. Each of them wants to buy a cassette player costing 320 rubles. Prove that at least one of them must wait for the next paycheck to make their purchase.

2. In a brigade of 7 people, the sum of the ages of the members is 332 years. Prove that three members can be chosen so that the sum of their ages is no less than 142 years.

§5. Number theory

Many wonderful applications of the Pigeon Hole Principle can be found in solving problems on divisibility properties of integers.

1. Prove that there exist two powers of two which differ by a multiple of 1987.

2. Prove that of any 52 integers, two can always be found such that the difference of their squares is divisible by 100.

3. Fifteen boys gathered 100 nuts. Prove that some pair of boys gathered an identical number of nuts.

A few more challenging problems for interested students:

Problem 1. Integers are placed in each entry of a 10×10 table, with no two neighbouring integers differing by more than 5 (two integers are considered neighbours if their squares share a common edge). Prove that two of the integers must be equal.

Problem 2. Prove that among any six people there are either three people, each of whom knows the other two, or three people, each of whom does not know the other two.

Problem 3. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.

Problem 4. A warehouse contains 200 boots of size 41, 200 boots of size 42, and 200 boots of size 43. Of these 600 boots, there are 300 left boots and 300 right boots. Prove that one can find among these boots at least 100 usable pairs.

Problem 5. The alphabet of a certain language contains 22 consonants and 11 vowels. Any string of these letters is a word in this language, so long as no two consonants are together and no letter is used twice. The alphabet is divided into 6 (non-empty) subsets. Prove that the letters in at least one of these groups form a word in the language.

Problem 6. Prove that we can choose a subset of a set of ten given integers, such that their sum is divisible by 10.

Problem 7. Given 11 different natural numbers, none greater than 20. Prove that two of these can be chosen, one of which divides the other.

Problem 8. Eleven students have formed five study groups in a summer camp. Prove that two students can be found, say A and B, such that every study group which includes student A also includes student B.

Problem 9. Each box in a 3×3 arrangement of boxes is filled with one of the numbers $-1, 0, 1$. Prove that of the eight possible sums along the rows, the columns, and the diagonals, two sums must be equal.

Problem 10. The digits $1, 2, \ldots, 9$ are divided into three groups. Prove that the product of the numbers in one of the groups must exceed 71.

Students should keep in mind that even if a problem seems difficult at first, it is always worthwhile to return to it later with fresh ideas. Avoid the temptation to flip straight to the solutions! And remember, some problems may also have alternative solutions that do not rely on the Pigeon Hole Principle.