IIIT Delhi RAM Maths Circle
Session 2
Handshakes, Networks & the Power of Counting

An Introduction to the Handshake Lemma

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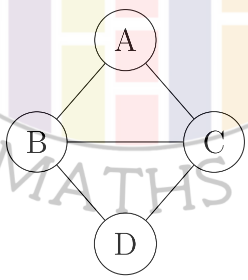
What Will We Discover?

What can a handshake tell us about a friendship network? Can we prove that a particular road network can never exist without drawing all the roads? Can odd and even numbers reveal something hidden about a group of people?
In this session, we will explore these questions using graphs and the Handshake Lemma. We will count connections, discover patterns, prove impossibilities, and solve puzzles involving friendships, roads, networks, and even a famous dinner-party mystery!
Most importantly, we will discover a powerful lesson: **Sometimes, changing the way we count can completely change the way we see a problem.**

What is a Graph?

A **graph** is a collection of points and connections between those points. For example, suppose four students are friends with one another. We can represent the students by dots and their friendships by lines joining the dots.



The dots represent the objects we are interested in, and the lines represent the connections between them.

Vertex

Each point in a graph is called a **vertex**. For example, in the graph above, A, B, C and D are vertices.

$\boxed{\text{Person or object} \longrightarrow \text{Vertex}}$

Edge

A line connecting two vertices is called an **edge**. For example, if two people shake hands, we can represent their handshake by an edge joining their two vertices.

$$\boxed{\text{Connection or handshake} \longrightarrow \text{Edge}}$$

Degree

The **degree** of a vertex is the number of edges connected to that vertex. For example, in the graph above, vertex B is connected to A, C and D. Therefore,

$$\boxed{\deg(B) = 3}$$

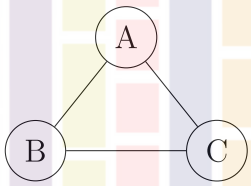
Similarly, we can find the degree of every other vertex by counting the edges connected to it. So remember:

$$\boxed{\text{Degree} = \text{number of connections of a vertex}}$$

For example, if two people shake hands, we can draw an edge between the two corresponding vertices.

$$\boxed{\text{Handshake} \longrightarrow \text{Edge}}$$

Thus, a group of people and their handshakes can be represented by dots and lines.



Here, there are 3 vertices and 3 edges.

1. Let's Count!

Problem 1 In a party, 4 people are present. Every person shakes hands with every other person exactly once.

(a) Draw a graph representing the situation.

(b) How many vertices does the graph have?

(c) How many edges does the graph have?

(d) What is the degree of each vertex?

If there are 5 people and every person shakes hands with every other person exactly once, how many handshakes take place? What happens when there are 6 people?

Number of people	Number of handshakes
2	
3	
4	
5	
6	
10	

Can you find a formula for the number of handshakes when there are n people?

2. From Degrees to the Handshake Lemma

Suppose that there are **5 people** at a party: A, B, C, D, and E.

- Person A shakes hands with **3 people**.
- Person B shakes hands with **2 people**.
- Person C shakes hands with **4 people**.
- Person D shakes hands with **1 person**.
- Person E shakes hands with **2 people**.

So, the number of handshakes made by the five people is

$$3, \quad 2, \quad 4, \quad 1, \quad 2.$$

Now represent each person by a **vertex** and each handshake by an **edge**. The **degree of a vertex** is the number of edges connected to that vertex. In other words, it tells us **how many handshakes that person made**. Therefore, the degrees of the five vertices are

$$3, \quad 2, \quad 4, \quad 1, \quad 2.$$

The sum of the degrees is

$$3 + 2 + 4 + 1 + 2 = 12.$$

But there were only 6 actual handshakes. Why?
Every handshake has two people involved. Therefore every handshake contributes exactly 2 to the total of the degrees. Hence,

$$3 + 2 + 4 + 1 + 2 = 2 \times (\text{number of handshakes}).$$

Therefore,

$$12 = 2 \times 6.$$

This gives us the important rule:

$$\boxed{\sum_v \deg(v) = 2E}$$

where $\deg(v)$ is the degree of vertex v and E is the number of edges. This is called the **Handshake Lemma**. The main idea is:

$$\boxed{\text{Every edge has exactly two ends.}}$$

Therefore, when we add all the degrees, every edge is counted exactly twice.

Problem 2 A town has 7 intersections. Each intersection is connected by roads to exactly 3 other intersections. Can such a road network exist? Explain.

Problem 3 There are 15 students in a group. Each student has either 2 or 3 friends among these 15 students. In other words, all the friendships we are considering are between students in this group. Prove that the number of students who have exactly 3 friends in this group is even.

Problem 4 Seventeen people attend a meeting. Each person shakes hands with at least one other person, and no two people shake hands with each other more than once.

(a) Show that at least two people must have shaken hands with the same number of people.

(b) Explain why it is impossible for every person to have shaken hands with exactly 9 other people.

Problem 5 A graph has 10 vertices. Can every vertex have odd degree? Explain your answer.

Problem 6 Four married couples attend a dinner party, so there are 8 people in total. Some people shake hands with others, but no one shakes hands with their own spouse or with themselves. At the end of the night, the host asks the other 7 people how many hands they shook. To his surprise, every single person gives a different number. How many hands did the host's spouse shake?

Problem 7 2n students are sitting around a circular table. Each student is friends with the two students sitting directly next to them. Some students may also be friends with students across the table. Prove that the number of students who have an odd total number of friends is even.

Problem 8 In a town, there are n people, where $n \geq 3$. It is known that every person has an odd number of friends in the town. Prove that the town can be divided into two rooms such that every person has an even number of friends in their own room.

Problem 9 There are 6 people in a group. We are considering only the friendships between these 6 people. The number of friends each person has within this group of 6 people is:

$$2, \quad 3, \quad 3, \quad 4, \quad 4, \quad 5.$$

Can such a group of 6 people actually exist? Explain your answer.

Problem 10 At a school dance, there are 12 boys and 12 girls. Every dance is between one boy and one girl. No pair of students dances with each other more than once. Each boy dances with exactly 4 different girls.

(a) Is it possible for the 12 girls to have danced with different numbers of boys? Explain your answer.

(b) Suppose instead that every girl danced with the same number of boys. How many boys did each girl dance with?

One Final Thought

"You don't always need to draw everything to count it."
The Handshake Lemma shows us that by counting connections in the right way, we can discover patterns that are invisible at first.
That's the power of mathematical thinking!