Raising A Mathematician Foundation In association with KREA university

RAM - Maths Circle

Date: December 07, 2025
Krea University, TTK Road

1. Things that work like numbers

(We developed these ideas in this class.)

In earlier sessions, we saw how we **made** numbers out of thin air (actually not even that) by declaring properties of numbers -

1. 0 is a number
2. Every number n has a *successor* denoted by Sn.

… which gives us the whole numbers 0, $S0$, $SS0$ and so on. We can give more convenient names to these, but that is basically it. We then described $+$ as -

1. $0 + n = n + 0 = n$
2. $(Sm) + n = m + (Sn)$

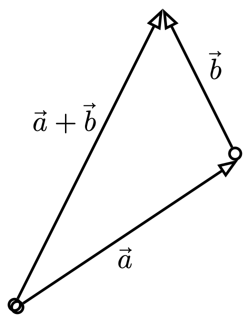
… and then went on to multiplication. $m \times 0 = 0 \times m = 0$ and $m \times (Sn) = m + m \times n$. From this, we can see that multiplication distributes over addition. (Can you prove it step by step?)

Question: What if we start from $+$ and $\times$ over some *things*? What *things* admit such notions of $+$ and $\times$ which meet -

1. $a + b = b + a$
2. $a + 0 = a$ (i.e. we define 0 as the "additive identity")
3. $a \times b = b \times a$
4. $a \times (b + c) = a \times b + a \times c$

1.1. Addition of "arrows" (a.k.a. "vectors")

Let's take arrows on a 2D plane (i.e. paper). We define addition of arrows like this -



i.e. to add two arrows $\vec{a}$ and $\vec{b}$, you touch arrange the arrows (keeping their length and direction intact), such that the tail of $\vec{b}$ touches the head of $\vec{a}$. Then the arrow from the tail of $\vec{a}$ to the head of $\vec{b}$ is what we call $\vec{a} + \vec{b}$.

Given that we're permitting arrows to be moved about as long as their length and direction are maintained, this means $\vec{a} + \vec{b} = \vec{b} + \vec{a}$. (Draw this and convince yourself about it.)

Also the zero-arrow is simply a point where the tail and the head are at the same place. So such a "zero-arrow" has zero length and therefore no particular direction.

We'll further defined the scaling of an arrow by a factor α (where α is a real number) and we'll write it as $\alpha\vec{a}$. If $\alpha < 0$, then the direction of the arrow is taken to be reversed. This also gives us a way to "subtract" vectors by defining subtraction by $\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$.

1.2. Multiplication of arrows

Now what would it mean to "multiply" two such arrows? Here is one possibility - We can describe an arrow using two numbers - its "length" and its "direction", given by the angle which it makes with the horizontal direction.

So what would $\vec{a} \times \vec{b}$ mean? If the length of $\vec{a}$ is L_a (without the vector sign) and its angle is θ_a, then what if we take $\vec{a} \times \vec{b}$ to mean "Scale (i.e. multiply) the length of $\vec{b}$ by L_a and rotate it by θ_a in the anti-clockwise direction[1]

It is easy to see that with this definition, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$, since in either case the length of the result vector is the **product of the lengths** of the two vectors being multiplied, and the angle of the result vector is the **sum of their angles**. For both those operations, the order doesn't matter.

Homework: Show that $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ also holds for this definition of arrow multiplication.

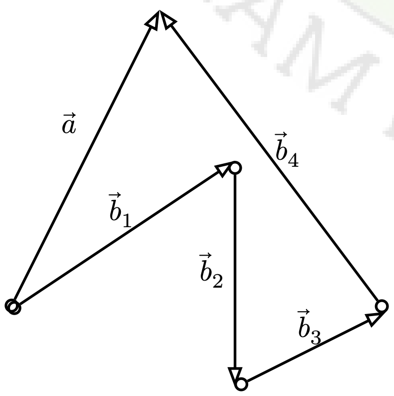
1.3. Next steps

So the "multiplicative identity" for the above definition is the arrow of length 1 that is pointing in the horizontal direction (i.e. angle = 0°). We'll write this as $\vec{1}$.

Supposing we consider the arrow of length 1 pointing vertically upward (i.e. at 90° to $\vec{1}$) and call it $\vec{\imath}$. Multiplying any arrow by $\vec{\imath}$ will therefore rotate the arrow counter-clockwise by 90° without changing its length.

Notice that $\vec{1} \times \vec{\imath} = \vec{\imath} \times \vec{1} = \vec{\imath}$ and $\vec{\imath} \times \vec{\imath} = -\vec{1}$, since $\vec{\imath} \times \vec{\imath}$ means "rotate $\vec{\imath}$ counter-clockwise by 90° which is an arrow pointing in the direction opposite to $\vec{1}$, which we write $-\vec{1}$.

One observation of our arrow addition rule is that it implies it doesn't matter what path we take from the tail of an arrow to the head of an arrow, the arrow remains the same.



In the above picture, no matter how we draw the $\vec{b}$ arrows, $\vec{a} = \vec{b}_1 + \vec{b}_2 + \vec{b}_3 + \vec{b}_4$ as long as the arrows flow in sequence from the tail of $\vec{a}$ to its head.

[1]You can pick clockwise too, but anti-clockwise is a convention and whatever you pick you have to stick with it and can't change your mind about what it means later on.

Therefore we can always express any arrow in the plane in the form $a\vec{1} + b\vec{\imath}$ for real numbers a and b.

Homework: Show that $\left(a\vec{1} + b\vec{\imath}\right) \times \left(c\vec{1} + d\vec{\imath}\right) = (ac - bd)\vec{1} + (ad + bc)\vec{\imath}$.

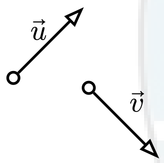
Homework: Show that the length l of an arrow $\vec{a} = \alpha\vec{1} + \beta\vec{\imath}$, is given by $l^2 = \alpha^2 + \beta^2 = \vec{a} \times \vec{a}^*$ where we define $\vec{a}^* = \alpha\vec{1} - \beta\vec{\imath}$.

Since vectors of the form $a\vec{1}$ are only scaling other vectors when multiplied, and we already have real numbers doing that job for us, we can simply drop $\vec{1}$ and write $\vec{a} = \alpha + \beta\vec{\imath}$.

Yes. Our “arrows” are the famous “complex numbers”!

2. Problems

1. Using the addition rule for 2D vectors (i.e. arrows in the 2D plane), show that the sum of interior angles of an n-gon (i.e. an n sided polygon) is $180° \times (n - 2)$. Does your argument work for convex and non-convex polygons?
2. Describe the shape you get when you plot $x = 4 + 2a, y = 4 - 2a$ on graph paper for all (real) values of a. Can you express this set of points using arrow addition rule? … where $\vec{r} = (x, y)$ is the arrow from the origin ($(0, 0)$) to a point (x, y) on the 2D plane.
3. Show using arrow logic, why $a - (-b) = a + b$ for ordinary real numbers a and b.
4. Line1 passes through the two points $(2, 3)$ and $(3, 2)$. Line2 passes through the two points $(1, 1)$ and $(4, 4)$. Find the point where the two lines intersect. First do this visually, then do this using any method you know, then try to do this using "arrow logic".
5. Using the "arrow multiplication is scaling and rotation" approach, show the trigonometric identities -
   1. $\sin(A + 90°) = \cos(A)$
   2. $\cos(A + 90°) = -\sin(A)$
   3. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
   4. $\cos(A + B) = \cos A \cos B - \sin A \sin B$.
6. You're given two arrows $\vec{u}$ and $\vec{v}$ as shown below. Draw the following combinations $\vec{u} + 2\vec{v}$, $2\vec{u} - 3\vec{v}$, $2\vec{u} + 3\vec{v}$.



7. Two arrows are said to be **linearly dependent** if one of them is the other multipled by an ordinary real number. Conversely, they are said to be **linearly independent** if neither of them can be expressed as the other multiplied by a real number. Given two linearly independent arrows $\vec{u}$ and $\vec{v}$ on the 2D plane (i.e. $\vec{u} \neq \alpha\vec{v}$ for any real α), can you express any other arrow on the plane in the form $\alpha\vec{u} + \beta\vec{v}$ where α and β are real numbers?
8. You're given two linearly independent arrows $\vec{u}$ and $\vec{v}$. You also have $\vec{a} = 2\vec{u} + 3\vec{v}$ and $\vec{b} = 3\vec{u} + 2\vec{v}$. Show that $\vec{a}$ and $\vec{b}$ are also linearly independent. Express the arrow $\vec{c} = 4\vec{u} + \vec{v}$ in terms of $\vec{a}$ and $\vec{b}$ – i.e. find α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.