Krea - RAM - Maths Circle - Session 13

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18/01/2025

1 Overview

This session explored the Fibonacci sequence and its deep connection to the golden ratio φ. We investigated why sequences satisfying the Fibonacci recurrence all converge to the same ratio, discovered Fibonacci numbers hidden within Pascal's triangle, and derived the self-referential properties of φ.

2 The Fibonacci Sequence

The Fibonacci sequence is defined by:

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

The first several terms are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Problems

(a) Compute the ratios $\dfrac{F_{n+1}}{F_n}$ for $n = 1, 2, \ldots, 10$. What do you notice?

(b) The ratios alternate above and below a limiting value. Why might this be?

3 The Golden Ratio

Derivation

Suppose the ratio $\dfrac{F_{n+1}}{F_n}$ converges to some limit φ. Then for large n:

$$\frac{F_{n+1}}{F_n} \approx \varphi \quad \text{and} \quad \frac{F_n}{F_{n-1}} \approx \varphi$$

Using the recurrence $F_{n+1} = F_n + F_{n-1}$, divide both sides by F_n:

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{F_n/F_{n-1}}$$

Taking the limit:

$$\varphi = 1 + \frac{1}{\varphi}$$

Multiplying by φ:

$$\varphi^2 = \varphi + 1$$

This gives the quadratic $\varphi^2 - \varphi - 1 = 0$. By the quadratic formula:

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618033988\ldots$$

4 The Remarkable Property: $\varphi - 1 = \dfrac{1}{\varphi}$

From $\varphi^2 = \varphi + 1$, we can derive:

$$\varphi^2 - \varphi = 1 \implies \varphi(\varphi - 1) = 1 \implies \varphi - 1 = \frac{1}{\varphi}$$

This means:

$$\varphi = 1.618033988\ldots \quad \text{and} \quad \frac{1}{\varphi} = 0.618033988\ldots$$

The golden ratio and its reciprocal share the same decimal expansion! This is a unique property: φ is the only positive number x such that $x - 1 = \dfrac{1}{x}$.

5 Generalized Fibonacci Sequences

Consider any sequence (a_n) satisfying the same recurrence:

$$a_1 = p, \quad a_2 = q, \quad a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3$$

Claim: For any starting values $p, q > 0$, the ratio $\dfrac{a_{n+1}}{a_n} \to \varphi$.
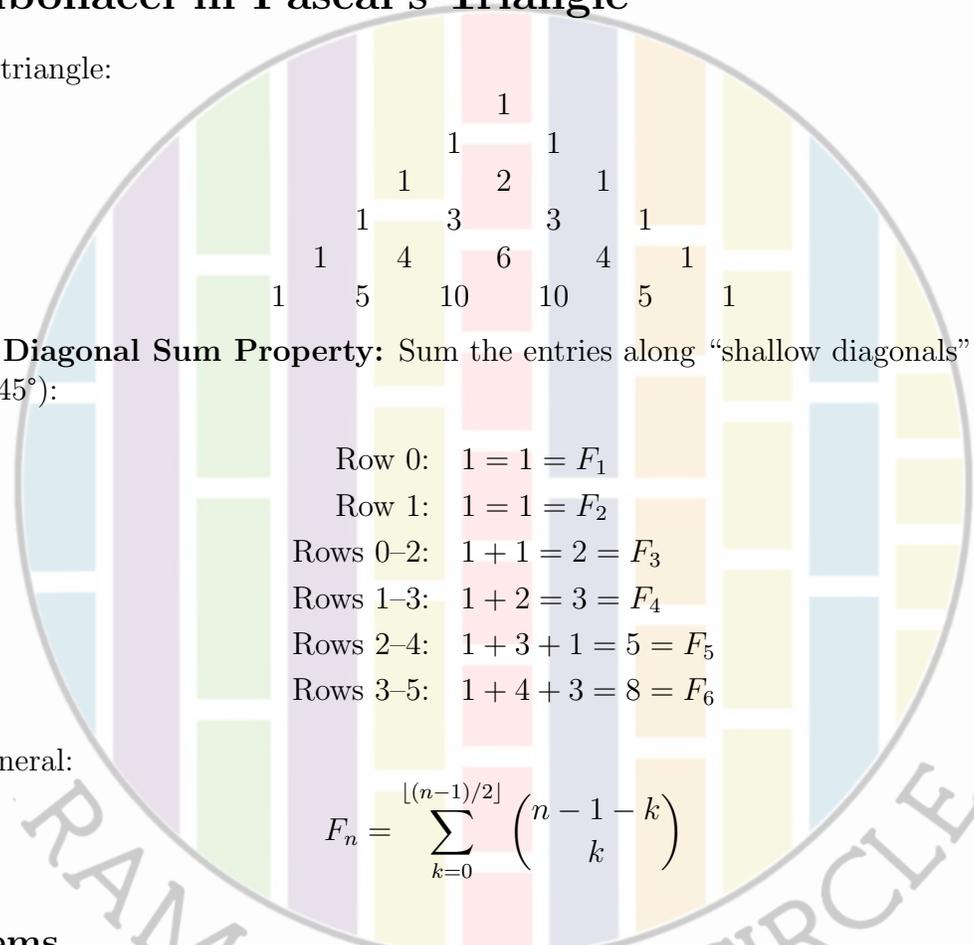
Examples

- **Lucas numbers:** $2, 1, 3, 4, 7, 11, 18, 29, \ldots$ (start with $p = 2, q = 1$)

- **Starting with 5, 8:** $5, 8, 13, 21, 34, \ldots$ (this is actually a shifted Fibonacci!)

- **Starting with 3, 7:** $3, 7, 10, 17, 27, 44, 71, \ldots$

Problems

(a) Compute the first 8 terms and the ratios for the sequence starting with $a_1 = 3, a_2 = 7$.

(b) Why does the starting value not affect the limiting ratio? (Hint: Think about what happens as $n \to \infty$.)

(c) What happens if we allow negative starting values? Does the ratio still converge to φ?

6   Fibonacci in Pascal's Triangle

Pascal's triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

The Diagonal Sum Property: Sum the entries along "shallow diagonals" (going up-right at 45°):

$$
\begin{array}{lll}
\text{Row 0:} & 1 = 1 = F_1 \\
\text{Row 1:} & 1 = 1 = F_2 \\
\text{Rows 0–2:} & 1 + 1 = 2 = F_3 \\
\text{Rows 1–3:} & 1 + 2 = 3 = F_4 \\
\text{Rows 2–4:} & 1 + 3 + 1 = 5 = F_5 \\
\text{Rows 3–5:} & 1 + 4 + 3 = 8 = F_6
\end{array}
$$

In general:

$$
F_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k}
$$

Problems

(a) Verify this formula for F_7 and F_8 by summing the appropriate diagonals.

(b) Why does this identity hold? (Hint: Use the Pascal recurrence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.)

(c) The identity can be written as $F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots$. Express F_{10} using this formula and verify.