RAM Math Circle - Chennai

Synopsis for August 24 2025

We began the discussion by reviewing ideas related to path counting on a rectangular array of boxes, which the students learned in the previous session on August 17, 2025. This time, they also explored a minor variant,computing shortest paths on a grid,which also leads to Pascal's Triangle.

Following that, we discussed how the solutions to several problems from the sessions on August 24 and August 31, 2025, relate to Catalan numbers and the Catalan triangle. More specifically, we constructed explicit "dictionaries" (bijective proofs) between the following problems:

- Counting shortest paths that do not cross the diagonal (Session report of August 31, 2025)
- The Bunny Hop problem (Session report of August 31, 2025)
- The Balanced Parentheses problem (Session report of August 31, 2025)
- The Non-crossing Handshake problem (Session report of August 31, 2025)

A dictionary between two problems provides a framework that shows how and why those problems are essentially equivalent. For example, consider the problems of counting shortest paths that do not cross the diagonal and the Bunny Hop problem. The dictionary between these is as follows:

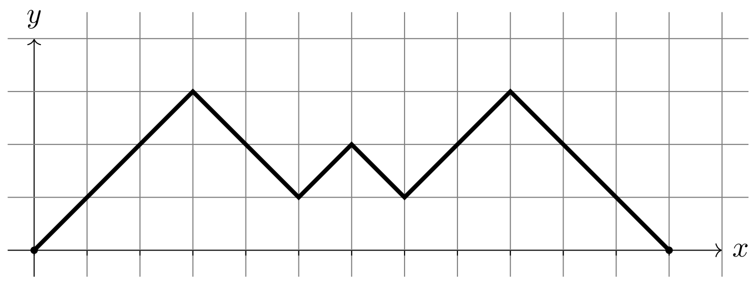
- Moving right on the grid corresponds to a bunny hopping up the stairs.
- Moving up on the grid corresponds to a bunny hopping down the stairs.
- The diagonal in the grid corresponds to the ground level in the Bunny Hop problem.
- Therefore:
  - The bunny not going below ground corresponds to a path not crossing the diagonal.
  - The bunny starting and returning to ground level corresponds to a path that starts and ends on the diagonal.

Under this dictionary, any direction sequence for path on the grid can be translated into a bunny hopping sequence and vice-versa. Similarly, we constructed analogous dictionaries between several other problem pairs. During the problem-solving session, we explored additional problems related to Catalan numbers. The goal now is to build dictionaries between each of these new problems and the earlier ones associated with Catalan numbers.

The explicit formula for the n^{th} Catalan number and the Catalan recurrence (see the challenging questions below) will be discussed in the upcoming session. The solutions to the remaining problems, along with a **complete master dictionary** connecting all these Catalan-related problems, will also be presented in future sessions. A few students expressed interest in constructing their own version of this "master dictionary," which we hope will deepen their understanding of the underlying combinatorial structures.

Additional Problems on Catalan numbers

Problem 1. **Dyck paths** A *Dyck path* of length $2n$ is a lattice path that starts at $(0,0)$, ends at $(2n,0)$, uses up-steps $U=(1,1)$ and down-steps $D=(1,-1)$, and never goes below the x–axis.



Example: A Dyck path (never dipping below the axis) that ends on the horizontal axis.

(Qn 1) For $n = 1$ and $n = 2$, list all Dyck paths (write them as words in U and D) and draw them on a grid.

(Qn 2) For $n = 3$, draw *all* Dyck paths of length 6. How many are there?

(Qn 3) Draw all Dyck paths of length 8 ($n = 4$). How many are there?

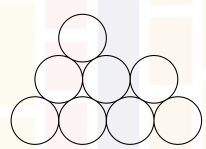
(Qn 4) **Challenging.** Let P be a Dyck path of length $2n$, and let $2k$ be the first time P returns to height 0 after the start. Show that P factors uniquely as

$$P \;=\; U\,A\,D\,B,$$

where A is a Dyck path of length $2(k-1)$ and B is a Dyck path of length $2(n-k)$. Deduce the Catalan recurrence

$$C_n \;=\; \sum_{k=1}^{n} C_{k-1}\,C_{n-k}, \qquad C_0 = 1.$$

Problem 2. Stacking coins in the plane. We place unit coins in the plane so that the bottom row has n consecutive, mutually touching coins on a horizontal line. Higher coins must be tangent to exactly two coins directly below (so they are "supported"), and no coins may overlap. Two stacks are considered the same if the sets of coin centers coincide.



Example: One valid stack when $n = 4$ (n= number of coins in bottom row).

(Qn 1) For $n = 1$ and $n = 2$, draw all possible stacks.

(Qn 2) For $n = 3$, draw *all* possible stacks. How many are there?

(Qn 3) Draw all stacks for $n = 4$.

(Qn 4) **Challenging.** How shall we deduce the Catalan recurrence? Let S_n be the number of stacks on n bottom coins. We want to show

$$S_n \;=\; \sum_{k=0}^{n-1} S_k\,S_{n-1-k}, \qquad S_0 = 1.$$

Problem 3. Ballot sequences (equal-vote, never-behind). Consider an election with two candidates A and B in which exactly $2n$ votes are cast: *precisely n votes for A and n votes for B.* Votes are tallied in order. Encode each vote for A by "+" and each vote for B by "−".

A *ballot sequence* of length $2n$ is a word with n pluses and n minuses such that every partial sum is nonnegative: if s_k is the number of $+$'s minus the number of $-$'s among the first k symbols, then $s_k \geq 0$ for all $k = 1, 2, \ldots, 2n$. Equivalently, candidate A is never behind candidate B at any time during the count.

Examples for small n:

$n = 1$: Valid: `+-` (i.e., AB). Invalid: `-+` (drops to -1 at the first step; A is behind immediately).

$n = 2$: Valid ballot sequences (and A/B versions):

$$\texttt{++--}\ (AABB), \qquad \texttt{+-+-}\ (ABAB).$$

Their partial sums are $1, 2, 1, 0$ and $1, 0, 1, 0$, respectively.

(Qn 1) For $n = 3$ and $n = 4$, list *all* ballot sequences (i.e., all valid orderings of n $+$'s and n $-$'s) and record the partial-sum sequence for each.

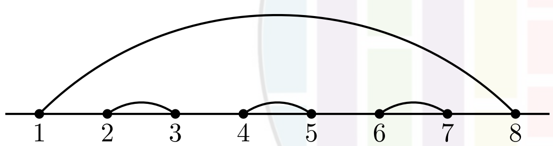
(Qn 2) **Challenging.** Let w be a ballot sequence of length $2n$, and let $2k$ be the first time its partial sum returns to 0 (its first "return"). Show that w decomposes uniquely as

$$w \;=\; +\,u\,-\,v\,,$$

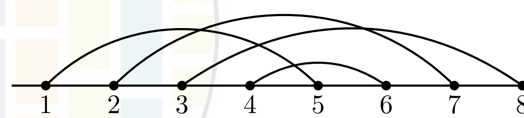
where u is a ballot sequence of length $2(k-1)$ and v is a ballot sequence of length $2(n-k)$. Deduce the Catalan recurrence

$$C_n \;=\; \sum_{k=1}^{n} C_{k-1}\, C_{n-k}\,, \qquad C_0 = 1.$$

Problem 4. Noncrossing matchings on 2n points. Place $2n$ labeled points $1, 2, \ldots, 2n$ on a horizontal line. A *(perfect) matching* is a collection of n disjoint arcs drawn above the line so that each point is incident to exactly one arc. A matching is *noncrossing* if no two arcs intersect.



Example: valid (noncrossing).



Example: invalid (arcs cross).

(Qn 1) Draw all noncrossing matchings on $2n$ points for $n = 1$ (2 points) and $n = 2$ (4 points).

(Qn 2) Draw all noncrossing matchings on $2n$ points for $n = 3$ (6 points). How many do you get?

(Qn 3) Draw all noncrossing matchings on $2n$ points for $n = 4$ (8 points).

(Qn 4) **Challenging.** Let M_n be the number of noncrossing matchings on $2n$ points. In any noncrossing matching, suppose point 1 is matched with point $2k$.

i. Show that the points $\{2, 3, \ldots, 2k-1\}$ (inside the arc $(1, 2k)$) form an independent noncrossing matching on $2(k-1)$ points.

ii. Show that the points $\{2k+1, \ldots, 2n\}$ (to the right) form an independent noncrossing matching on $2(n-k)$ points.

iii. Deduce the Catalan recurrence

$$M_n \;=\; \sum_{k=1}^{n} M_{k-1}\, M_{n-k}, \qquad M_0 = 1,$$

Problem 5. Cash–counter (ballot/Catalan). At a counter that sells only 50-paise tickets, $2n$ people stand in a queue. Exactly n of them have a 50-paise coin (no change needed) and the other n have a one-rupee coin (they need one 50-paise in change). The cashier starts with no money. How many ways can the $2n$ people line up so that the cashier never runs out of change?