Krea - RAM - Maths Circle - Session 12

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1 Overview

A problem session combining ideas from modular arithmetic and infinity concepts from previous sessions to ensure that students have understood the concepts discussed and to help them understand real life applications of the same.

2 Problem 1

There are n lights in a row, all initially off. You have switches $S_1, S_2, \ldots, S_n$ where switch S_k toggles all lights at positions divisible by k.

1. After pressing all switches once, which lights are on?
2. Explain the math behind the pattern we observe.
3. If there were infinitely many lights and switches, which lights would be on?

3 Problem 2

At 12:00, the minute and hour hands of a clock coincide perfectly.

1. At what exact time do they next coincide? Express your answer as a fraction.
2. How many times do the hands coincide in a 12-hour period?

4 Problem 3: Luhn's Algorithm (Credit Card Check Digits)

Every credit card number has a built-in error detection system invented by Hans Peter Luhn in 1954. Here's how it works:

1. **The Algorithm:** Starting from the rightmost digit, double every second digit. If doubling gives a number greater than 9, subtract 9. Add all digits together. The total must be divisible by 10.

2. **Example:** Check if 4539 1488 0343 6467 is valid.

   - Write the digits: 4 5 3 9 1 4 8 8 0 3 4 3 6 4 6 7
   - Mark alternating positions from the right: 4 $\underline{5}$ 3 $\underline{9}$ 1 $\underline{4}$ 8 $\underline{8}$ 0 $\underline{3}$ 4 $\underline{3}$ 6 $\underline{4}$ 6 $\underline{7}$
   - Double the underlined digits: 4 (10) 3 (18) 1 (8) 8 (16) 0 (6) 4 (6) 6 (8) 6 (14)
   - Subtract 9 from any result > 9: 4 1 3 9 1 8 8 7 0 6 4 6 6 8 6 5
   - Sum: $4+1+3+9+1+8+8+7+0+6+4+6+6+8+6+5 = 82$
   - Is 82 divisible by 10? No, so this number is invalid.

3. **Problems:**

   (a) Verify that 4532 0151 2421 5765 is a valid card number.

   (b) The card number 7992 7398 713? has one digit replaced by ?. Find the missing digit.

   (c) If you swap two adjacent digits in a valid card number, will the Luhn check always catch the error? Why or why not?

   (d) Why do you think the algorithm doubles every second digit instead of just adding all digits?

5 Problem 4: ISBN Codes (Book Identification)

Every book has a unique ISBN (International Standard Book Number). We'll look at two versions.

ISBN-10

A 10-digit code $d_1d_2d_3d_4d_5d_6d_7d_8d_9d_{10}$ where the last digit is chosen so that:

$$10d_1 + 9d_2 + 8d_3 + 7d_4 + 6d_5 + 5d_6 + 4d_7 + 3d_8 + 2d_9 + 1d_{10} \equiv 0 \pmod{11}$$

The check digit d_{10} can be 0–9 or X (representing 10).

(a) Verify that 0-306-40615-2 is a valid ISBN-10.

(b) The ISBN 0-13-110362-? is missing its check digit. Find it.

(c) If you swap two adjacent digits in a valid ISBN-10, will the check always catch it? (Compare to Luhn's algorithm.)

6 Take-Home Problem

Try to come up with more checking systems similar to ISBN codes and Luhn's algorithm that help capture a wider range of errors without being too computationally expensive.