**IIIT Delhi - RAM Maths Circle**
**Session 3**

(Organized by the Department of Mathematics, IIIT Delhi)

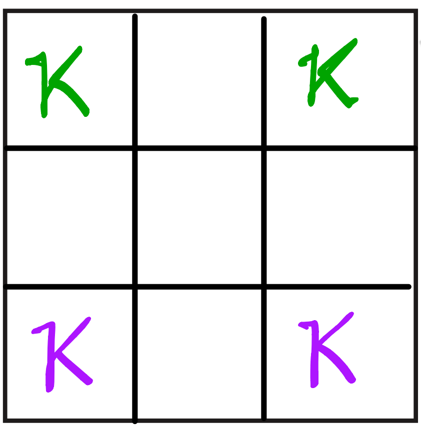
IIIT-Delhi September 28th, 2025

**Please attempt the problems with * once you are done with the other problems.**

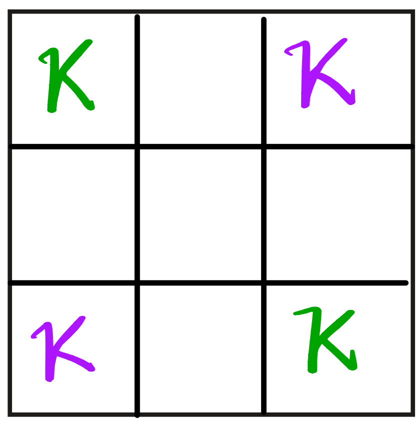
## 1 The Concept of a Graph

**Problem 1.** Cosmic liaisons are established among the nine planets of the solar system. Rockets travel along the following routes: Earth-Mercury, Pluto–Venus, Earth-Pluto, Pluto-Mercury, Mercury-Venus, Uranus-Neptune, Neptune-Saturn, Saturn-Jupiter, Jupiter-Mars, and Mars-Uranus. Can a traveler get from Earth to Mars?

**Problem 2.** Several knights are situated on a $3 \times 3$ chessboard as shown in Figure 1.1. Can they move, using the usual knight's move, to the position shown in Figure 1.2?



1.1

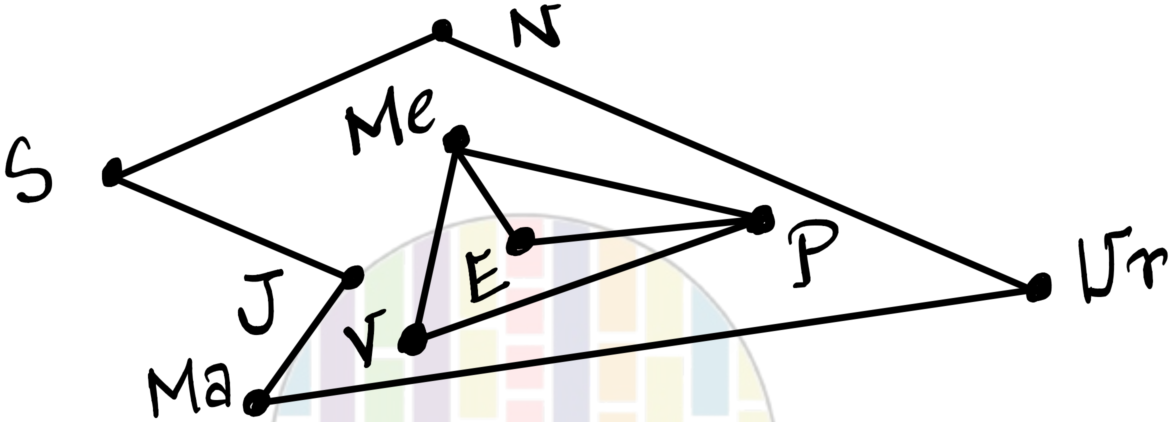


1.2

**Problem 3***. A chessboard has the form of a cross, obtained from a $4 \times 4$ chessboard by deleting the corner squares. Can a knight travel around this board, passing through each square exactly once and ending on the same square?

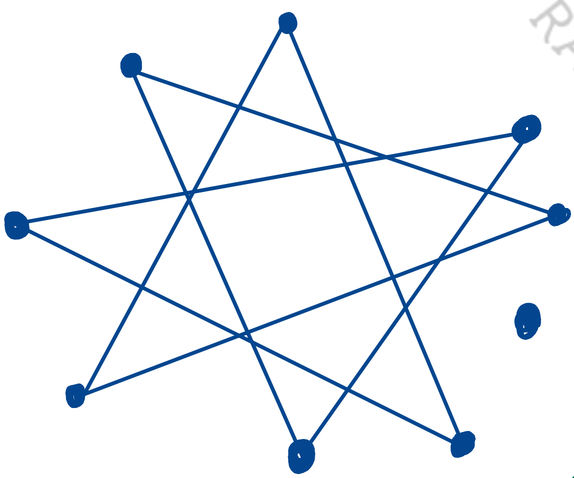
**Problem 4.** In a country, there are nine cities labeled 1–9. Two cities are connected by an airplane route if and only if the two-digit number formed by naming one city, then the other, is divisible by 3. Can a traveler get from City 1 to City 9?

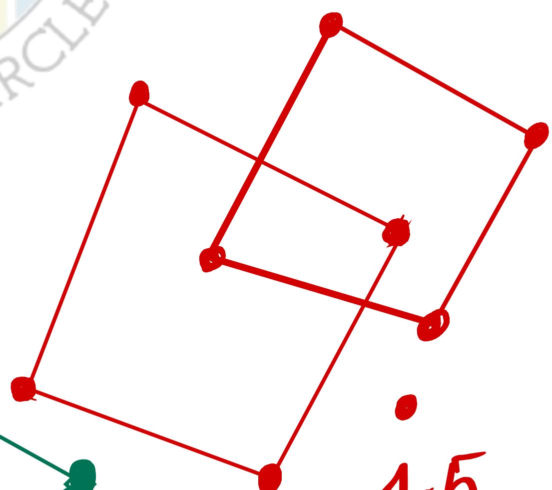
Note that the same graph can be represented in different ways. For example, the graph of Problem 1 can be redrawn as in Figure 1.3 below. **The only important thing about a graph is which vertices are connected and which are not**.

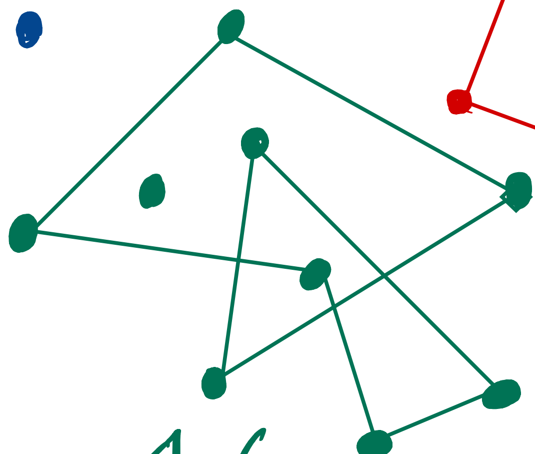



**Two graphs that are actually identical, though perhaps drawn differently, are called** *isomorphic*.

**Problem 5.** Find, in Figures 1.4, 1.5, and 1.6, a graph isomorphic to the graph of Problem 2 (Figure 1.1, 1.2).

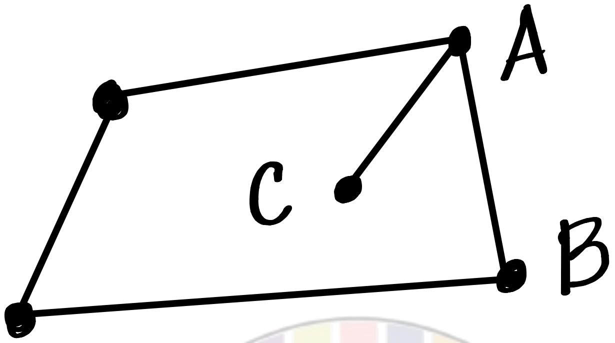

1·4


1·5


1·6

# 2 The Degree of a Vertex: Counting the Edges

The *degree* of a vertex is the number of edges that start at that vertex. In the graph of Figure 2.1, vertex $A$ has degree 3, vertex $B$ has degree 2, and vertex $C$ has degree 1.




**Problem 6.** In Smallville there are 15 telephones. Can they be connected by wires so that each telephone is connected with exactly five others?

**Counting rule.** *The number of edges in a graph equals half the sum of the degrees of all its vertices.*

**Problem 7.** In a kingdom there are 100 cities, and four roads lead out of each city. How many roads are there altogether?

Notice that the sum of the degrees of all the vertices in a graph must be even (otherwise dividing by 2 would not give an integer number of edges).

**Definition.** A vertex of odd degree is an *odd vertex*; one of even degree is an *even vertex.*

**Theorem.** The number of odd vertices in any graph is even.

**Proof.** Prove it!

**Problem 8.** There are 30 students in a class. Can it happen that 9 of them have 3 friends each, 11 have 4 friends each, and 10 have 5 friends each?

**Problem 9.** In Smallville there are 15 telephones. Can they be connected so that (a) each telephone is connected with exactly 7 others; (b) there are 4 telephones each connected to 3 others, 8 telephones each to 6 others, and 3 telephones each to 5 others?

**Problem** 10*. Can 9 line segments be drawn in the plane, each intersecting exactly 3 others?

**Problem** 11*. John, coming home from Disneyland, said he saw an enchanted lake with 7 islands, to each of which there led either 1, 3, or 5 bridges. Is it true that at least one of these bridges must lead to the shore of the lake?