

1 Mathematical detectives

The central idea of this worksheet was the distinction between:

- observing a pattern,
- collecting evidence,
- finding a counterexample, and
- giving an argument that covers every case.

The overarching aim was to encourage students to approach mathematics as an investigative activity. Rather than presenting statements as facts to be proved, the session invited students to act as “detectives”: gather evidence, question their conclusions, look for exceptions, and ask what would constitute a convincing mathematical argument.

The aim is not to introduce formal proof techniques as a collection of rules. Instead, the students should gradually *feel the need for proof*: examples can make a statement believable, but they do not explain why it must continue to work.

The worksheet deliberately moves from familiar statements to patterns, then to a geometric explanation, and finally to algebraic arguments.

Aims and objectives

The session aimed to help students:

- distinguish between an *example*, a *pattern*, a *conjecture*, a *counterexample*, and a *proof*;
- understand that checking several examples provides evidence but does not, in general, prove a statement about every case;
- recognize that a single counterexample is sufficient to disprove a universal statement;
- develop the habit of actively searching for counterexamples rather than simply accumulating confirming examples;
- formulate conjectures from numerical and geometric patterns;
- understand the need for an argument that works for an arbitrary case;
- experience both geometric and algebraic ways of explaining a pattern; and
- begin to see proof not merely as a formal exercise, but as an explanation of why a mathematical phenomenon must occur.

The investigations

The session began with simple statements whose truth values were investigated through examples. These included

- the sum of two odd numbers being even,
- the claim that the sum of two prime numbers is always even,
- the square of an odd number being odd,
- the product of two consecutive positive integers being even.

These examples provided an opportunity to discuss why a statement can look convincing after several successful tests while still requiring further justification. Students considered explicitly whether checking 20 examples would be enough to establish a statement as true, leading to a discussion of the limitations of experimental evidence and of what it means to establish a claim for *all* cases.

The next investigation introduced *counterexamples*. Students searched for examples that disproved claims such as “Every prime number is odd,” “Every number divisible by 4 is also divisible by 8,” and “The sum of three consecutive numbers is always even.” They also considered the claim that if $a^2 > b^2$, then $a > b$, and the geometric claim that every quadrilateral with four equal sides is a square. These problems illustrated that counterexamples can arise in different mathematical settings and that one carefully chosen example can overturn a universal claim.

The second major part of the session focused on *patterns and conjectures*. Students investigated

$$1, \quad 1 + 3, \quad 1 + 3 + 5, \quad 1 + 3 + 5 + 7, \dots$$

and observed the square-number pattern

$$1 + 3 + \dots + (2n - 1) = n^2.$$

The students were encouraged to distinguish between the evidence supplied by examples and a proof of the general statement. A conversation between two detectives reinforced this distinction: checking several cases gives evidence, but it does not rule out failure at a later value of n .

Students then explored a *geometric explanation* of the odd-number pattern. Starting with a 1×1 square, each successive odd number of dots can be arranged to enlarge the existing square. In general, an $n \times n$ square requires

$$n + (n + 1) = 2n + 1$$

additional dots to become an $(n + 1) \times (n + 1)$ square. Since $2n + 1$ is precisely the next odd number, the construction explains why

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

holds in general.

Finally, students investigated the sequence of square numbers

$$1, 4, 9, 16, 25, 36, \dots$$

and examined the differences between consecutive terms:

$$3, 5, 7, 9, 11, \dots$$

They first identified the pattern experimentally and then explained it algebraically using

$$(n + 1)^2 - n^2 = 2n + 1.$$

This connected the two investigations: the same odd numbers that build successive squares are also the differences between consecutive square numbers.

