Synopsis for December 14 2025

We continued our math session by having session in two separate classrooms for the junior and senior batches. However, both groups followed the same continuation theme of number theory from the previous session.

Junior Batch:

The junior batch focused on strengthening their understanding of basic concepts in number theory. They then worked on exercises involving prime factorisation and finding the HCF and LCM of small numbers using factor trees Emphasis was placed on step-by-step reasoning and clarity of method. Students actively participated and asked questions during problem-solving.

We began the session by revisiting the material covered in the previous two classes. We recalled the definitions of prime numbers and composite numbers and discussed some important edge cases. In particular, we thought about whether -1 can be considered a prime number, whether the definition of prime numbers should be restricted to positive integers and why, and whether zero can be classified as a composite number. These questions helped students think more carefully about the role of definitions in mathematics.

Next, we recalled our earlier exercise of listing all prime numbers less than 1000. We also discussed pairs of prime numbers with specific differences, such as twin primes (primes with difference 2) and pairs of primes with difference 4. This naturally led to a revision of the concept of co-prime numbers and what it means for two numbers to have no common factors other than 1.

The session then moved on to the idea of HCF (also called GCD). We discussed the meaning of the highest common factor and reviewed different methods for finding it. Since prime factorisation was already familiar to most students, we practised factoring numbers and used this method to find the HCF of pairs such as 90 and 24, and 336 and 1746.

While examining these prime factisations, students noticed an important pattern: every number can be written as a product of prime numbers in a unique way. From this observation, we were guided to state the Fundamental Theorem of Arithmetic, which was then written down formally as follows:

Every integer greater than 1 can be expressed as a product of prime numbers, and this factorisation is unique, except for the order in which the prime factors are written.

Using the same examples, students were then introduced to Euclid's algorithm. We learned how to carry out the algorithm step by step and discussed why such an algorithm is useful. In particular, we reflected on the advantages of Euclid's algorithm over prime factorisation, especially when dealing with large numbers.

To illustrate this point clearly, we applied Euclid's algorithm to a problem involving much larger numbers, namely 654312 and 4356. Through this example, students observed how efficient and practical the algorithm is when compared to prime factorisation for large integers. Overall, the students actively participated and showed good engagement with both computational and conceptual aspects of the topic.

Senior Batch

The senior batch continued with the theme of number theory from the previous session, but at a more conceptual and proof-oriented level. The session began with a brief recall of the following concepts:

- Definition of a prime number.
- Prime factorisation of a positive integer.

Students were asked to recall examples and properties, and to explain why prime factorisation plays a central role in number theory. A discussion was held on why the number 1 is not considered a prime. Students were guided to observe that if 1 were treated as a prime, then the uniqueness of prime factorisation would fail. This motivated the need for excluding 1 from the set of prime numbers. The **Fundamental Theorem of Arithmetic** was then stated and discussed.

Every integer greater than 1 can be written as a product of prime numbers, and this representation is unique up to the order of the factors.

Examples were used to illustrate both existence and uniqueness. The importance of this theorem in defining and working with HCF and LCM was also emphasised.

Euclid's Proof of the Infinitude of Primes

Euclid's proof that there are infinitely many prime numbers was presented in full detail.

Proof: Assume, for contradiction, that there are only finitely many prime numbers. Let these primes be

$$p_1, p_2, p_3, \ldots, p_k.$$

Consider the number

$$N = p_1 p_2 p_3 \cdots p_k + 1.$$

This number is greater than 1. When we divide N by any prime p_i, the remainder is 1. Hence, none of the primes $p_1, p_2, \ldots, p_k$ divides N.

Therefore, either N itself is prime, or it has a prime factor not among the listed primes. In both cases, we obtain a prime number not in the original list, which contradicts the assumption that all primes were already listed. Hence, there are infinitely many prime numbers. □

Students observed that the product of the first k prime numbers is not itself a prime number. They checked examples and found counterexamples. However, it was noted that the number

$$p_1 p_2 \cdots p_k + 1$$

always has a prime factor that is larger than p_k. This observation was connected back to Euclid's proof and helped reinforce the idea of infinitude of primes.

Sieve of Eratosthenes

The Sieve of Eratosthenes was introduced as a systematic method to list all prime numbers less than a given bound, such as 1000. The idea of repeatedly removing multiples of primes was explained, and students discussed how this method can be used not only to list primes but also to count them.

Counting Idea: Inclusion–Exclusion Principle

To understand how counting works in the sieve method, the idea of inclusion–exclusion was introduced.

A Venn diagram was drawn for the case of three sets, representing numbers divisible by 2, 3, and 5. Students observed how overlaps must be added and subtracted correctly.

It was noted that drawing Venn diagrams becomes difficult for four or more sets, motivating an algebraic approach. The formula for the size of the union of four sets A_1, A_2, A_3, A_4 was written explicitly:

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = &\sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \\ &- |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

Similarly, the general pattern for five sets was discussed, highlighting the alternating signs. The idea of proving this formula using induction was briefly outlined.

Derangement Problem

To further illustrate the idea of inclusion–exclusion, the classical derangement problem for five letters was discussed. An **arrangement** of five letters means any permutation of the letters. For example, if the letters are

$$A, B, C, D, E,$$

then

$$(A, B, C, D, E) \quad \text{and} \quad (B, A, C, D, E)$$

are both arrangements. A **derangement** is an arrangement in which no letter appears in its original position. For example,

$$(B, C, D, E, A)$$

is a derangement, while

$$(A, C, B, D, E)$$

is **not** a derangement because the letter A is in its correct position. Thus, a **non-derangement** is an arrangement in which at least one letter is fixed in its original position. Problem is to count the number of derangements. Let the total number of arrangements of five letters be

$$5! = 120.$$

Define the following sets:

- A_1: all arrangements in which letter A is in the correct position.
- A_2: all arrangements in which letter B is in the correct position.
- A_3: all arrangements in which letter C is in the correct position.
- A_4: all arrangements in which letter D is in the correct position.
- A_5: all arrangements in which letter E is in the correct position.

Then,

$$|A_i| = 4! = 24 \quad \text{for each } i,$$

since fixing one letter leaves 4 letters free to permute.

- For any two distinct indices $i \neq j$, $|A_i \cap A_j| = 3! = 6$, since two letters are fixed and three remain free.
- For any three distinct indices, $|A_i \cap A_j \cap A_k| = 2! = 2$.
- For any four distinct indices, $|A_i \cap A_j \cap A_k \cap A_l| = 1! = 1$.
- For all five letters fixed, $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 1$.

The number of arrangements with at least one letter fixed is:

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = & \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \\ & - \sum |A_i \cap A_j \cap A_k \cap A_l| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|. \\ = & \, 5 \times 24 - \binom{5}{2} \times 6 + \binom{5}{3} \times 2 \\ & - \binom{5}{4} \times 1 + \binom{5}{5} \times 1.\end{aligned}$$

$$= 120 - 60 + 20 - 5 + 1 = 76.$$

The number of derangements is obtained by subtracting the above from the total number of arrangements:

$$5! - 76 = 120 - 76 = 44.$$

This example also demonstrated how inclusion–exclusion can be used to count complex objects by systematically correcting over-counting.

Application to Counting Prime Numbers

Finally, the counting problem was phrased explicitly:

How many positive integers less than 1000 are not divisible by any prime less than or equal to 31?

Students discussed how the inclusion–exclusion principle can be applied to count numbers divisible by 2, 3, 5, 7, and other primes. Once we count all numbers that are divisible by at least one prime less than or equal to 31 using the inclusion–exclusion principle, the remaining numbers correspond to prime numbers less than 1000.

For example, let A_1 denote the set of all multiples of 2 less than 1000, excluding the number 2 itself. Then

$$|A_1| = 500 - 1,$$

since 2 must be excluded as it is not a composite number.

Similarly, let A_2 denote the set of all multiples of 3 less than 1000, excluding 3 itself. Then

$$|A_2| = 333 - 1,$$

as 3 is also a prime number and should not be counted as composite. The same idea applies to multiples of other primes.

Although there are 11 prime numbers less than or equal to 31 that need to be considered, students found that counting the multiples of each prime individually was straightforward. However, they observed that writing down all the terms in the inclusion–exclusion formula involved too many summands and became difficult to manage.

Some students quickly noticed that any intersection of five such sets must be empty, because there is no number less than 1000 that is a multiple of five distinct primes. For example,

$$2 \times 3 \times 5 \times 7 \times 11 = 2310 > 1000,$$

and even

$$3 \times 5 \times 7 \times 11 = 1155 > 1000.$$

This implies that any non-empty intersection of four sets must include 2 as one of its prime factors.

Using this observation, students listed all possible intersections involving four primes, three primes, and two primes. Based on these calculations, they attempted to arrive at the number 168, which is the number of prime numbers less than 1000.

Although the class did not complete the entire computation within the available time, the students clearly understood the strategy and the steps involved. They worked in small groups and divided the calculations among themselves, which encouraged parallel thinking. This collaborative approach helped build mutual trust and strategy development, which are key aspects of a math circle session.