

1 Sequences

A *sequence* of natural numbers is a list of numbers that follow a certain pattern. That pattern may not be known beforehand, but often times we are interested in finding out what that pattern is.

*Sometimes we can find a nice formula to describe each element in the sequence based on its position in the sequence but without calculating the earlier elements of the sequence. Such a formula is called a **closed-form formula**.*

*Other times it can be easier to find a formula for the next element in the sequence in terms of the previous elements. Such a formula is called a **recurrence relation**.*

Some recurrence relations can be used to find a closed-form formula for a sequence. But this may not always be possible.

Generally, the following notation is used.

Suppose we have a sequence $1, 3, 5, 7, 9, \dots$. Then the first element is 1, so let's call it a_1 . Let's denote the second element by a_2 , third element by a_3 and so on. In general, a_n will denote the n^{th} element of the sequence. So the number in the suffix (which is called the *index*) denotes the position of an element in a sequence. In the above example, $a_2 = 3, a_3 = 5, a_4 = 7, a_5 = 9$, and so on. We can make a guess that a_6 must be 11, a_7 must be 13, and in general a_n must be $2n - 1$. This is a closed-form formula for our sequence because it is in terms of the index.

Suppose you are given the formula $a_n = n(n + 1)$, then you can generate the corresponding sequence of numbers: $1(1 + 1), 2(2 + 1), 3(3 + 1), 4(4 + 1), 5(5 + 1), \dots$ which is same as

$$2, 6, 12, 20, 30, \dots$$

You can also directly find any term in the sequence. For example, the 100^{th} term in the above sequence will be $a_{100} = 100(100 + 1) = 10100$.

If you are given a *recursive formula* such as

$$a_n = 2a_{n-1} + a_{n-2}, \quad a_0 = a_1 = 1$$

then we can again find the sequence as follows:

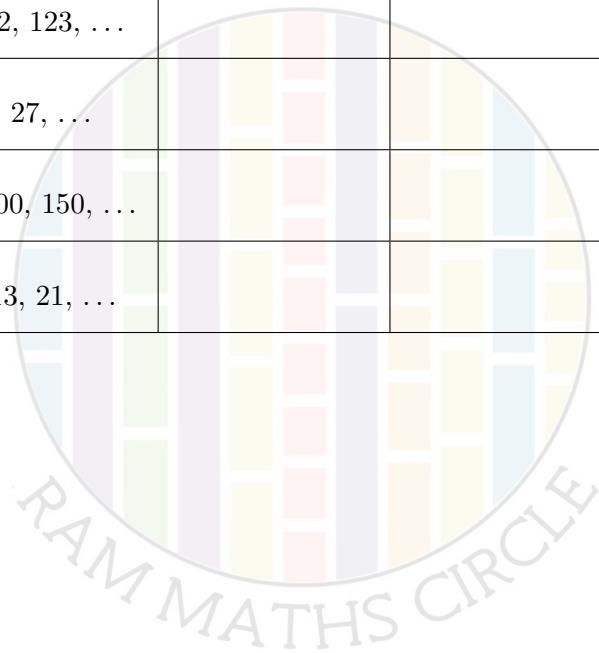
$$1, 2, 5, 12, 29, \dots$$

However, in this case we cannot directly find an element in any given position. For example, to find the element in the 100^{th} position, we need to know the elements in the 99^{th} and 98^{th} positions first.

1.1 Examples

Guess the next number in each of the given sequences and think of a possible formula to describe the next element of each sequence:

Sequence a_n	Next term	Possible formula
2, 2, 2, 2, 2, ...		
5, -5, 5, -5, 5, -5, ...		
1, 5, 2, 10, 3, 15, ...		
4, 8, 12, 16, 20, 24, ...		
-2, -1, 6, 25, 62, 123, ...		
3, 6, 11, 18, 27, ...		
0, 10, 30, 60, 100, 150, ...		
1, 2, 3, 5, 8, 13, 21, ...		



1.2 Some families of sequences

1.2.1 Arithmetic sequences

If the terms of a sequence differ by a constant, we say the sequence is arithmetic. If the initial term of the sequence is a and the common difference is d , then we have

- Recursive formula: $a_n = a_{n-1} + d, a_0 = a.$
- Closed form formula: $a_n = a + nd$

Adding up terms of an arithmetic sequence

Suppose the first n terms of an arithmetic sequence are $\{a, a + d, a + 2d, \dots, a + (n - 1)d\}$, so that $a_{n-1} = a + (n - 1)d$. Lets us denote the sum of these terms by S . We can express S in two ways:

$$\begin{aligned} S &= a + (a + d) + (a + 2d) + \dots + (a + (n - 1)d) \\ S &= (a + (n - 1)d) + (a + (n - 2)d) \dots + (a + d) + a \end{aligned}$$

Adding up the two expressions, which are identical but order-reversed, we can say that

$$2S = n(2a + (n - 1)d)$$

Therefore the required sum is $S = \frac{n(2a + (n - 1)d)}{2}$.

Note: If you choose to label the terms of your sequence starting with index 1, then the last term of the sequence will be $a_n = a + (n - 1)d$. This does not affect the formula for S which is the sum of the first n terms of the sequence.

1.2.2 Geometric sequences

A sequence is called geometric if the ratio between successive terms is constant. Suppose the initial term is a and the common ratio is r then we have,

- Recursive formula: $a_n = ra_{n-1}, a_0 = a.$
- Closed form formula: $a_n = ar^n$

Adding up terms of a geometric sequence

Suppose the terms of a geometric sequence are $\{a, ar, ar^2, \dots, ar^n\}$. Lets us denote the sum of these terms by S . Then

$$\begin{aligned} S &= a + ar + ar^2 + \dots + ar^n \\ rS &= ar + ar^2 + \dots + ar^{n+1} \end{aligned}$$

Can you complete the argument to find a formula for S ?

If you choose to label the terms of your geometric sequence starting with index 1, then the last term of the sequence will be _____ and the sum will be $S =$ _____.

Arithmetic sequences model real life scenarios in which a quantity increases (or decreases) steadily with a constant difference. Geometric sequences can be used to model situations in which the ratio of consecutive terms is constant, that is there is a constant multiplicative factor involved.

1.3 More examples

1. Find recursive and closed formulae for the following sequences:
 - (a) 2, 7, 12, 17, 22, 27, 32, ...
 - (b) 2, 6, 18, 54, 162, 486, ...
 - (c) 9, 18, 36, 72, 144, 288, ...
 - (d) 100, 94, 88, 82, 76, 70, ...
 - (e) 6, 10, 18, 34, 66, ...

2. There are 20 rows of seats on a concert hall: 25 seats are in the 1st row, 27 seats on the 2nd row, 29 seats on the 3rd row, and so on. If the price per ticket is Rs.32, how much will be the total sales for a one-night concert if all seats are taken?

3. Suppose one person tells 3 new friends about a rumour (stage 1), each of those 3 people tell 3 new people (stage 2) and so on. How many people will have heard of the rumour by stage 10?

4. Suppose a bacterial culture doubles every hour. If an experiment starts with 20 bacteria in a petri dish, how many will there be in 24 hours?

5. Find a recurrence relation for the number a_n of ways to distribute n distinct objects into 5 boxes.

6. Suppose on January 1 you deposit Rs.20 in an empty piggy bank. On Jan. 8 you deposit Rs. 30; on Jan. 15 you deposit Rs.40; and each week thereafter you deposit Rs.10 more than the previous week. What amount will you deposit in the 52nd week?

7. Starting with any rectangle, we can create a new, larger rectangle by attaching a square to the longer side. For example, if we start with a 2×3 rectangle, we can attach a 3×3 square, forming a 3×5 rectangle; then we can attach a 5×5 square to make a 8×5 rectangle and so on.
 - (a) Create a sequence of rectangles using this rule starting with a 1×2 rectangle. Then write out the sequence of perimeters for the rectangles (the first term of the sequence would be 6, since the perimeter of a rectangle is 6 - the next term would be 10).
 - (b) Repeat the above part this time starting with a 1×3 rectangle.
 - (c) Find recursive formulas for each of the sequences of perimeters you found in parts (a) and (b). Give the initial conditions as well.
 - (d) Are the sequences arithmetic? Geometric? If not, are they close to being either of these (i.e., are the differences or ratios almost constant)? Explain.

8. Find the number of ways in which you can climb 100 steps if you can go up 1 or 2 steps at a time.