Krea - RAM - Maths Circle - Session 11

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1 Overview

An exploration of different sizes of infinity, building from listing fractions to discovering that some infinities are bigger than others.

2 Can We List All the Fractions?

1. **The Challenge: Listing Fractions in Order**
   - We started by asking: can we make a list of all positive fractions (like $\frac{1}{2}, \frac{3}{4}, \frac{7}{5}$) so that every fraction appears exactly once?
   - At first this seems impossible since between any two fractions there are infinitely many more!
   - Students attempted various approaches: listing by size (failed), listing all halves then thirds (never finishes one group).
2. **The Zigzag Method**
   - We arranged fractions in a grid: row 1 has fractions with numerator 1, row 2 has numerator 2, etc.
   - By zigzagging diagonally through this grid, we can reach every fraction eventually.
   - This shows that fractions can be matched one-to-one with counting numbers (1, 2, 3, 4, ...).
   - Key insight: even though fractions seem "more dense," they are the same "size" of infinity as the counting numbers.

3 Are All Infinities the Same Size?

1. **Attempting to List All Decimal Numbers Between 0 and 1**
   - We asked: can we also list all decimal numbers (like $0.5, 0.333\ldots, 0.7182818\ldots$) between 0 and 1?

- Students tried various listing strategies, none seemed to capture everything.

2. **Cantor's Clever Argument**

- Suppose someone claims they have a complete list of all decimals between 0 and 1.
- We showed how to always build a number that is NOT on their list:
  - Look at the 1st digit of the 1st number, pick something different for your new number's 1st digit.
  - Look at the 2nd digit of the 2nd number, pick something different for your new number's 2nd digit.
  - Continue forever down the diagonal.
- The new number differs from every number on the list (it disagrees with the nth number in the nth digit).
- Therefore, no list can ever be complete, thus, the decimals are a "bigger" infinity!

3. **What This Means**

- Not all infinities are equal. Some collections are genuinely larger than others.
- The counting numbers and fractions are the "smallest" infinity (called countable).
- The decimal numbers form a "bigger" infinity (called uncountable).
- This was one of the most surprising discoveries in mathematics.